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TETRAD FIELDS AND GRAVITATIONAL FIELDS

BY

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CONTENTS

Synopsis

The consequences of the hypothesis – recently advanced by C. Møller – that gravitation may be described by a tetrad field, are examined by studying the most general Lagrangian obtained as a linear combination of invariants. A particular choice of these can be made on the basis of correspondence with the Newtonian theory. The field equations obtained from this particular Lagrangian, although somewhat different from those of Møller, give rise, in the case of a static, spherically symmetric system, to the usual Schwarzschild metric. Moreover, in the cases considered explicitly by Møller the solutions of the field equations are the same.

A conserved energy-momentum complex having the property that the energy density is localizable is also derived. The use of a tetrad field to describe the structure of space-time allows the introduction of spinor fields, and in particular of the neutrino field, in a natural way. A new coupling between fermions and gravitation also follows from this theory.

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I. Introduction

The concepts of energy and momentum and of conservation laws, which have played a very important part in all physics, have some peculiar features in the theory of general relativity. Owing to the general covariance of the theory, there exist an infinite number of conservation laws, all equally valid[1].

The selection from these of what may be called the "energy-momentum" conservation law is essentially a matter of physical interpretation. Various "complexes", each with certain definite properties, have therefore been proposed[2]. In particular, MØLLER[3] introduced some conditions which must be satisfied in order that a complex be the energy-momentum complex. Especially he required that the energy density be localizable, i. e. a scalar under the group of purely spatial coordinate transformations, and that the total energy and momentum be transformed like a four-vector with respect to the Lorentz group. Subsequently MØLLER[4] was able to show that no complex satisfying both these conditions can be formed within the framework of Einstein's theory.

To come out of this situation, he proposed a new formulation of the theory [4] [5], in which the fundamental variables of the gravitational field were assumed to be the 16 components of a tetrad field, connected by ten relations with the metric tensor.

There are two possible approaches to tetrad fields. The usual one consists in regarding additional degrees of freedom associated with tetrads as non-physical. In the framework of this philosophy tetrads may be used as working tools in the same way as one uses potentials in electrodynamics. However, similarly as in electrodynamics, where physically meaningful quantities have to be gauge-invariant, the entire physical content of the theory must, in the usual approach to tetrads in general relativity, stay invariant with respect to arbitrary Lorentz rotations of tetrads (which may change from point to point). When using tetrads in this spirit, one remains strictly on the level of orthodox general relativity.*

The second approach to tetrads, which is closer in spirit to that advocated by MØLLER, is based on the hypothesis that *all* 16 degrees of freedom of tetrads may be physically meaningful; here one demands of the theory that its physical quantities are invariant only with respect to *constant* tetrad rotations.

* The question how convenient tetrads are in the treatment of *global* conserved quantities (and in other problems of the usual theory) is discussed by one of us (J. P.) in Proceedings of the Warsaw Conference on Relativistic Theories of Gravity.

Once this assumption is made, it is possible to obtain an energy-momentum complex satisfying both conditions. Since, for a given physical situation, the determination of the tetrad field requires sixteen equations, Møller added to the usual Einstein equations a set of six equations $\varphi_{\alpha\beta} = 0$, where $\varphi_{\alpha\beta}$ is a skew tensor function of the tetrads and their derivatives. He was then able to show that, in the linearized case, his equations

$$\left.\begin{aligned} G_{\alpha\beta} &= -\chi T_{\alpha\beta} \\ \varphi_{\alpha\beta} &= 0 \end{aligned}\right\} \quad (I.1)$$

are equivalent to the Einstein weak-field equations, and that the solution of (I.1) for a static, spherically symmetric system gives rise to the usual Schwarzschild metric. The new theory is thus in agreement with all known experimental facts. However, the general expression for $\varphi_{\alpha\beta}$ contains a certain arbitrariness, which it would seem interesting to try to eliminate by attempting a variational-principle formulation of the Møller theory. This is what we will try to do in the present paper.

In other words, we intend to investigate the theory that follows from these assumptions: (1) the Møller hypothesis that all 16 degrees of freedom of tetrads are physically meaningful (invariance only with respect to constant tetrad rotations); (2) that a canonical formulation of the theory in terms of an action principle is possible.

We see that in this way we investigate a theory that from a heuristic point of view is wider than orthodox general relativity, which so beautifully solves all problems meaningful in the framework of its philosophy. Nevertheless we are of opinion that a generalization of this type is worthy of investigation.

It is of importance to realize that even the dynamical laws determining in this theory the metric tensor may be slightly different from the usual Einstein equations; nevertheless the usual philosophy of general relativity (including the principle of equivalence) here remains the same.

Some misunderstanding may arise in connection with the possibility of Fernparallelismus in this theory, which might be taken to be contradictory to the orthodox interpretation of general relativity. We would point out that when, in the usual theory, there exists a physical vectorial field, e. g. potentials of the vectorial mesonic field, the notion of Fernparallelismus to that vector may be introduced without violation of any first principles of the theory. When one understands tetrads as four physical vectorial fields fixed by consistent dynamical laws, and the metric as a secondary concept defined in terms of these fields, there is no need to change the usual interpretation. There is, however, the possibility that the dynamical laws governing tetrads are different from the usual ones, which determine tetrads only up to x-dependent Lorentz rotations.

The chief purpose of this paper is precisely to investigate the possibilities associated with the small latitude that can be allowed in the choice of the dynamics determining tetrad quantities. The interpretation of the theory, however, stays the same as in orthodox general relativity.

In the next section the tetrad formalism is briefly introduced, and our notations fixed. In section III the general structure of the Lagrangian $\mathfrak{L}$ of the tetrad field is studied. The actual construction of the most general $\mathfrak{L}$ by means of the invariants formed of tetrads is made in section IV.

As noted by Møller, the space-time continuum used in this theory differs from the usual Riemannian space by the existence of a tetrad in every point. This fact has interesting consequences in geometry; for instance it allows the introduction of the concept of absolute parallelism of two vectors at distant points.

For the definition of absolute parallelism and its developments, such as the absolute derivation, the reader is referred to the work of MØLLER (5). Here we only want to point out that the most important geometrical notion in this space is that of torsion, characterized by a tensor $\Lambda_{[\alpha\beta]\gamma}$ such that when $\Lambda_{[\alpha\beta]\gamma} = 0$, space-time is flat. In accordance with this the torsion tensor is used in section IV to build up the invariants. The Lagrangian is written as a linear combination of four of these invariants; hence it will contain four arbitrary constants; these are determined in section V, which deals with the linearized form of the theory. The result is that if we choose two constants equal to zero, the linearized theory is equivalent to that of Einstein.

Since we assume this to be a necessary condition, we obtain in this way a Lagrangian $\mathfrak{L}$ depending on two constants k_1 and k_2 only; the further development of the theory will be based on this $\mathfrak{L}$. From the linear approximation it also follows — which is interesting to note — that k_1^{-1} is equal to the Einstein gravitational constant. Although the constant k_2 will remain undetermined in this work, it seems possible that further developments will be able to give us some information on it.

Another characteristic of a theory of this type is that, together with the group of coordinate transformations, there is another group leaving the Lagrangian invariant, i. e. a simultaneous and equal rotation of all the tetrads. This is shown in section VI and is also used in appendix B to introduce spinors by means of the representations of these rotations. This simple and natural way of defining spinors has two noteworthy consequences: (a) the derivative of a spinor can be consistently defined as the partial derivative; (b) since the theory is invariant only with respect to the group of proper tetrad rotations, the neutrino has its own place in the theory, and its existence is related to the fundamental structure of space-time.

The field equations are derived from the Lagrangian in section VII. They turn out to be different from those proposed by Møller. This is due to the fact that the possibility of using the Levi-Civita tensor in the construction of $\varphi_{\alpha\beta}$ was overlooked, whereas the same tensor plays a fundamental part in this work. The field equations derived in section VII are

$$\left.\begin{aligned} k_1\, G_{\alpha\beta} + k_2\, F_{<\alpha\beta>} &= -\, T^{(b)}{}_{\alpha\beta} - T^{(f)}{}_{<\alpha\beta>} \\ k_2\, F_{[\alpha\beta]} &= -\, T^{(f)}{}_{[\alpha\beta]}, \end{aligned}\right\} \quad (\mathrm{I.2})$$

where again $G_{\alpha\beta}$ is the Einstein tensor and $F_{\alpha\beta}$ is analogous to $\varphi_{\alpha\beta}$. In our case, anyway $F_{\alpha\beta}$ has a symmetric as well as an antisymmetric part. The matter tensor has been divided, in (I.2), into a boson part and a fermion part. The existence of the new "skew" equation $k_2 F_{[\alpha\beta]} = - T^{(f)}_{[\alpha\beta]}$ is interpreted to mean that the space-time structure is determined by matter not only through the symmetric energy-momentum tensor, but also through the skew part $T^{(f)}_{[\alpha\beta]}$ which is connected with the spin angular-momentum tensor. All the physical content of the theory now becomes clearer: the usual Riemannian space-time is not general enough to describe classical matter as well as spinor fields; this deficiency is shown in the lack of a satisfactory (in the Møller sense) energy-momentum complex and of a natural way to introduce spinors [(6)].

If we assume that the influence of a boson and that of a fermion on the space-time structure are qualitatively different, we need a scheme wider than that of Einstein, and this might be a space with a built-in tetrad lattice. For the theory to be in agreement with experimental facts it is further necessary that the tetrad field obtained by solving eqs. (I.2) in the case of a static, spherically symmetric system, gives rise to the Schwarzschild metric. That it is so, is shown in section VIII. It is interesting that this particular tetrad field is just identical with that obtained by Møller [(5)] for the same case.

Finally, in section IX, the energy-momentum conservation law is discussed. It is shown that the energy-momentum complex we derive satisfies both Møller conditions. In addition to a conservation law $\mathsf{T}_{\alpha}{}^{\beta}{}_{,\beta} = 0$, a tensorial conservation law is seen to follow from the structure of the field equations. In general, $\mathsf{T}_{\alpha}{}^{\beta}$ can be written in the same form as the Møller energy-momemtum complex:

$$\mathsf{T}_{\alpha}{}^{\beta} = \Theta_{\alpha}{}^{\beta} + \mathfrak{U}_{\mu}^{[\beta\nu]} \Delta^{\mu}{}_{\alpha\nu}, \tag{I.3}$$

$$\mathsf{T}_{\alpha}{}^{\beta} = \mathfrak{U}_{\alpha}^{[\beta\gamma]}{}_{,\gamma}, \tag{I.4}$$

and the only non-tensorial term in (I.3) is $\Delta^{\mu}{}_{\alpha\nu}$. Both the superpotential $\mathfrak{U}_{\alpha}^{[\beta\gamma]}$ and $\mathsf{T}_{\alpha}{}^{\beta}$ can be written as a linear combination of the corresponding Møller term and a new one:

$$\mathsf{T}_{\alpha}{}^{\beta} = \mathsf{T}_{\alpha}{}^{\beta}{}_{(M)} + \mathsf{T}'_{\alpha}{}^{\beta}.$$

In the case of a static, spherically symmetric system the new term vanishes, while in the linearized case it can be written as a divergence and is moreover symmetric.

The use of tetrads in general relativity was already proposed by Einstein in 1928 [(7)]. He tried, however, to use the six new degrees of freedom contained in the tetrad field to describe electromagnetism. The equations derived by Einstein were shown to be incompatible with the Schwarzschild solution [(8)].

II. Preliminaries and Notations

Let V_4 be a normal hyperbolic Riemannian space characterized, in arbitrary coordinates x^{α}, by the symmetric metric tensor $g_{\alpha\beta}(x)$. Greek indices take the values 0, 1, 2, 3. The signature $(+,-,-,-)$ is assumed for the metric. The indices labelling the vectors of a tetrad will also be denoted by Greek letters, but with a "roof" above them, so that the index $\hat{\alpha}$ takes the values $\hat{0}, \hat{1}, \hat{2}, \hat{3}$. The Einstein summation convention is assumed.

Now, because of the normal character of V_4, one may always construct at a given point four vectors orthogonal to each other, one of them being time-like and normalized to plus one, the remaining ones space-like and normalized to minus one. Denoting the covariant components of these vectors by $g^{\hat{\alpha}}_{\alpha}(x)$, we may summarize their orthonormality properties in

$$g^{\alpha\beta}\, g^{\hat{\alpha}}_{\alpha}\, g^{\hat{\beta}}_{\beta} = g^{\hat{\alpha}\hat{\beta}}, \tag{II.1}$$

where $g^{\hat{\alpha}\hat{\beta}}$ is the numerical matrix of special relativity,

$$g^{\hat{\alpha}\hat{\beta}} = g_{\hat{\alpha}\hat{\beta}} = \begin{vmatrix} 1 & & & \\ & -1 & 0 & \\ & & -1 & \\ 0 & & & -1 \end{vmatrix}. \tag{II.2}$$

Note that the existence of such vectors is equivalent to Hilbert's condition. We will also assume that the $g^{\hat{\alpha}}_{\alpha}(x)$ as functions of x are differentiable, i. e. that they change from point to point in a regular way. The normal and roofed indices may be raised or lowered by means of the quantities $g_{\alpha\beta}$, $g^{\alpha\beta}$, $g_{\hat{\alpha}\hat{\beta}}$, $g^{\hat{\alpha}\hat{\beta}}$ respectively. (II.1) may now be written as

$$g^{\hat{\alpha}}_{\varrho}\, g^{\varrho}_{\hat{\beta}} = \delta^{\hat{\alpha}}_{\hat{\beta}}. \tag{II.3}$$

Multiplying (II.3) by four arbitrary scalars $A_{\hat{\alpha}}$ and by $g^{\hat{\beta}}_{\sigma}$, one obtains

$$\left(A_{\hat{\alpha}}\, g^{\hat{\alpha}}_{\varrho}\right) g^{\hat{\beta}}_{\sigma}\, g^{\varrho}_{\hat{\beta}} = A_{\hat{\alpha}}\, g^{\hat{\alpha}}_{\sigma},$$

and from this follows

$$g^{\hat{\beta}}_{\sigma}\, g^{\varrho}_{\hat{\beta}} = \delta^{\varrho}_{\sigma}, \tag{II.4}$$

$$g^{\hat{\beta}}_{\varrho}\, g_{\hat{\beta}\sigma} = g_{\varrho\sigma}. \tag{II.5}$$

Eq. (II.5) is the fundamental relation between the ten components of the metric tensor and the sixteen components of tetrads. While the tetrads completely determine the metric, the reverse is not true.

A Lorentz transformation of the roofed indices, i. e. a tetrad rotation

$$g'^{\hat{\alpha}}_{\alpha}(x) = L^{\hat{\alpha}}{}_{\hat{\beta}}(x)\, g^{\hat{\beta}}{}_{\alpha}(x) \tag{II.6}$$

with

$$g_{\hat{\varrho}\hat{\sigma}}\, L^{\hat{\varrho}}{}_{\hat{\alpha}}(x)\, L^{\hat{\sigma}}{}_{\hat{\beta}}(x) = g_{\hat{\alpha}\hat{\beta}}, \tag{II.7}$$

leaves the metric tensor invariant.

Adopting the notation

$$\left.\begin{aligned} g_{..} &= \det|g_{\alpha\beta}|, & g^{..} &= \det|g^{\alpha\beta}|, \\ g^{\hat{:}} &= \det\left|g^{\hat{\alpha}}_{\beta}\right|, & g^{:}_{\hat{}} &= \det\left|g^{\alpha}_{\hat{\beta}}\right|, \end{aligned}\right\} \quad \text{(II.8)}$$

one obtains

$$\left.\begin{aligned} g^{\hat{:}}\, g^{:}_{\hat{}} &= 1 \\ g_{..} &= -(g^{\hat{:}})^2 \\ g^{..} &= -(g^{:}_{\hat{}})^2. \end{aligned}\right\} \quad \text{(II.9)}$$

Therefore, if one understands as $\sqrt{-g_{..}}$ the positive branch of $\sqrt{\phantom{-g_{..}}}$, one gets

$$\sqrt{-g_{..}} = |g^{\hat{:}}|. \quad \text{(II.10)}$$

We conclude this section with the remark that, considering all metric quantities as known, one may attribute to a vector A four different kinds of component: A^{α}, A_{α}, $A^{\hat{\alpha}}$, $A_{\hat{\alpha}}$. The table below explains how one of them can be expressed through the others.

TABLE 1

$$\begin{aligned} A^{\alpha} &= A^{\alpha} && = g^{\alpha\beta} A_{\beta} = g^{\alpha}_{\hat{\beta}}\, A^{\hat{\beta}} = g^{\alpha\hat{\beta}} A_{\hat{\beta}} \\ A_{\alpha} &= g_{\alpha\beta} A^{\beta} = && A_{\alpha} = g_{\alpha\hat{\beta}} A^{\hat{\beta}} = g^{\hat{\beta}}_{\alpha}\, A_{\hat{\beta}} \\ A^{\hat{\alpha}} &= g^{\hat{\alpha}}_{\beta}\, A^{\beta} = g^{\hat{\alpha}\beta} A_{\beta} = && A^{\hat{\alpha}} = g^{\hat{\alpha}\hat{\beta}} A_{\hat{\beta}} \\ A_{\hat{\alpha}} &= g_{\hat{\alpha}\beta} A^{\beta} = g_{\hat{\alpha}}^{\ \beta} A_{\beta} = g_{\hat{\alpha}\hat{\beta}} A^{\hat{\beta}} = && A_{\hat{\alpha}}. \end{aligned}$$

III. The Lagrangian of the Tetrad Field

The principal aim of this work is the formulation of an action principle

$$\delta A = \delta \int \mathfrak{L}\,(g^{\hat{\alpha}}_{\alpha}(x),\ g^{\hat{\alpha}}_{\alpha,\beta}(x))\, d^4x = 0$$

for the tetrad field. We assume that the Lagrangian satisfies the following conditions:

(I) it must be a scalar (or pseudoscalar) density with respect to the group of coordinate transformations;

(II) it must be a scalar or pseudoscalar density with respect to the subgroup of constant Lorentz rotations of the tetrads, i. e. the rotations for which

$$\frac{\partial L^{\hat{\alpha}}{}_{\hat{\beta}}(x)}{\partial x^{\mu}} = 0;$$

(III) it must be a function of the $g^{\hat{\alpha}}_{\alpha}(x)$ and their first derivatives, and further it must be bilinear in the $g^{\hat{\alpha}}_{\alpha,\beta}(x)$.

This is sufficient to determine the general form of $\mathfrak{L}$. In fact, because of the covariance requirement, the first derivatives of the tetrads can appear in $\mathfrak{L}$ only in the form of covariant derivatives, i. e. as

$$g^{\hat{\alpha}}_{\alpha;\beta} = g^{\hat{\alpha}}_{\alpha,\beta} - \Gamma^{\varrho}_{\alpha\beta}\, g^{\hat{\alpha}}_{\varrho}. \tag{III.1}$$

Expressing the metric tensor in terms of tetrads and substituting in the Christofell symbol, we obtain the relation between the covariant and non-covariant derivatives of $g^{\hat{\alpha}}_{\alpha}$*

$$g_{\hat{\alpha}\beta;\gamma} = g^{\alpha}_{\hat{\alpha}}\, P^{[\nu\mu]\hat{\sigma}}_{[\alpha\beta]\gamma}\,(g_{\hat{\sigma}\nu,\mu} - g_{\hat{\sigma}\mu,\nu}), \tag{III.2}$$

where

$$4\, P^{[\nu\mu]\hat{\sigma}}_{[\alpha\beta]\gamma} = g^{\hat{\sigma}}_{\alpha}(\delta^{\nu}_{\beta}\delta^{\mu}_{\gamma} - \delta^{\nu}_{\gamma}\delta^{\mu}_{\beta}) + g^{\hat{\sigma}}_{\beta}(\delta^{\nu}_{\gamma}\delta^{\mu}_{\alpha} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\gamma}) + g^{\hat{\sigma}}_{\gamma}(\delta^{\nu}_{\beta}\delta^{\mu}_{\alpha} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}). \tag{III.3}$$

Therefore we can get a Lagrangian satisfying (I), (II) and (III) if we write it in the form

$$\mathfrak{L} = \frac{1}{4}\, g^{\frac{1}{2}}\, L^{[\alpha\beta][\mu\nu]}_{\hat{\gamma}\;\;\hat{\varrho}}\,(g^{\hat{\gamma}}_{\alpha,\beta} - g^{\hat{\gamma}}_{\beta,\alpha})\,(g^{\hat{\varrho}}_{\mu,\nu} - g^{\hat{\varrho}}_{\nu,\mu}). \tag{III.4}$$

The quantity $L^{[\alpha\beta][\mu\nu]}_{\hat{\gamma}\;\;\hat{\varrho}} = L^{[\mu\nu][\alpha\beta]}_{\hat{\varrho}\;\;\hat{\gamma}}$ must be a tensor with respect to coordinate transformations. It is clear that it can be constructed only from quantities like the tetrads and the Levi-Civita tensor.

Without specifying the form of $L^{[\alpha\beta][\mu\nu]}_{\hat{\gamma}\;\;\hat{\varrho}}$ it is already possible to draw some results from the form (III.4) of the Lagrangian. By varying the tetrad field in (III.4) we obtain the equations of motion

$$-2\left[g^{\frac{1}{2}}\, L^{[\lambda\delta][\mu\nu]}_{\hat{\lambda}\;\;\hat{\varrho}}\, g^{\hat{\varrho}}_{\mu,\nu}\right]_{,\delta} + \frac{\partial}{\partial g^{\hat{\lambda}}_{\lambda}}\left[g^{\frac{1}{2}}\, L^{[\alpha\beta][\mu\nu]}_{\hat{\gamma}\;\;\hat{\varrho}}\right]\cdot g^{\hat{\gamma}}_{\alpha;\beta}\, g^{\hat{\varrho}}_{\mu;\nu} = -g^{\frac{1}{2}}\, T^{\lambda}_{\hat{\lambda}}. \tag{III.5}$$

To derive (III.5) we have added to (III.4) the Lagrangian of an external field and have introduced the notation

$$g^{\frac{1}{2}}\, T^{\lambda}_{\hat{\lambda}} = \frac{\delta \mathfrak{L}_{\mathrm{ext.}}}{\delta g^{\hat{\lambda}}_{\lambda}}.$$

From (III.5) a conservation law follows immediately:

$$\left[g^{\frac{1}{2}}\, T^{\lambda}_{\hat{\lambda}} + \frac{\partial}{\partial g^{\hat{\lambda}}_{\lambda}}(g^{\frac{1}{2}}\, L^{[\alpha\beta][\mu\nu]}_{\hat{\gamma}\;\;\hat{\varrho}})\, g^{\hat{\gamma}}_{\alpha;\beta}\, g^{\hat{\gamma}}_{\mu;\nu}\right]_{,\lambda} = 0. \tag{III.6}$$

* By $[\alpha\beta]$ and $<\alpha\beta>$ we will indicate antisymmetry and symmetry with respect to the indices α, β.

Note that both terms in (III.6) are vector densities. Introducing the notation

$$g\hat{:}\, t^{\lambda}_{\hat{\lambda}} = \frac{\partial}{\partial g^{\hat{\lambda}}_{\lambda}} (g\hat{:}\, L^{[\alpha\beta]\,[\mu\nu]}_{\hat{\gamma}\quad\hat{\varrho}})\, g^{\hat{\gamma}}_{\alpha;\beta}\, g^{\hat{\varrho}}_{\mu;\nu} \tag{III.7}$$

$$\Theta^{\lambda}_{\hat{\lambda}} = t^{\lambda}_{\hat{\lambda}} + T^{\lambda}_{\hat{\lambda}}, \tag{III.8}$$

we may write (III.6) as

$$g\hat{:}\, \Theta^{\lambda}_{\hat{\lambda};\lambda} = 0, \tag{III.9}$$

which is a completely covariant equation. Furthermore, from (III.5) it is easily seen that

$$g\hat{:}\, \Theta^{\lambda}_{\hat{\lambda}} = g\hat{:}\, U^{[\lambda\delta]}_{\hat{\lambda};\delta} \tag{III.10}$$

$$g\hat{:}\, U^{[\lambda\delta]}_{\hat{\lambda}} = 2\, g\hat{:}\, L^{[\lambda\delta]\,[\mu\nu]}_{\hat{\lambda}\quad\hat{\varrho}}\, g^{\hat{\varrho}}_{\mu;\nu}. \tag{III.11}$$

These results will be used in section IX to obtain a conserved energy-momentum complex.

IV. Construction of the Lagrangian from the Invariants of the Tetrad Field

In section III the general structure of the Lagrangian has been studied. Now the most general form of $\mathfrak{L}$ satisfying our conditions will be written explicitly as a linear combination of the invariants bilinear in the first derivatives of the tetrad field.

In order to construct the invariants we introduce the tensors

$$\gamma_{[\alpha\beta]\gamma} = g^{\hat{\alpha}}_{\alpha}\, g_{\hat{\alpha}\beta;\gamma}, \tag{IV.1}$$

$$\Lambda_{[\alpha\beta]\gamma} = (g^{\hat{\alpha}}_{\alpha,\beta} - g^{\hat{\alpha}}_{\beta,\alpha})\, g_{\hat{\alpha}\gamma} \equiv (g^{\hat{\alpha}}_{\alpha;\beta} - g^{\hat{\alpha}}_{\beta;\alpha})\, g_{\hat{\alpha}\gamma}, \tag{IV.2}$$

$$\Phi_{\alpha} = \Lambda_{[\alpha\beta]}{}^{\beta} = \gamma^{\beta}{}_{\alpha\beta}, \tag{IV.3}$$

$$\check{\Lambda}^{\alpha} = \eta^{\alpha\beta\gamma\delta}\Lambda_{[\beta\gamma]\delta}, \tag{IV.4}$$

$$\check{\Lambda}_{\alpha} = \eta_{\alpha\beta\gamma\delta}\Lambda^{[\beta\gamma]\delta}, \tag{IV.5}$$

$$\check{\Lambda}^{[\varrho\sigma]}{}_{\tau} = \frac{i}{2}\eta^{\varrho\sigma\lambda\nu}\Lambda_{[\lambda\nu]\tau}, \tag{IV.6}$$

$$\check{\Lambda}_{[\varrho\sigma]\tau} = \frac{i}{2}\eta_{\varrho\sigma\lambda\nu}\Lambda^{[\lambda\nu]}{}_{\tau}, \tag{IV.7}$$

where the tensor η is defined by

$$\eta^{\alpha\beta\gamma\delta} = -g^{\hat{:}}\,\varepsilon^{\alpha\beta\gamma\delta}$$
$$\eta_{\alpha\beta\gamma\delta} = \;\; g^{\hat{:}}\,\varepsilon_{\alpha\beta\gamma\delta}$$

and $\varepsilon^{\alpha\beta\gamma\delta}$, $\varepsilon_{\alpha\beta\gamma\delta}$ are the numerical Levi-Civita symbols.

$\Lambda_{[\alpha\beta]\gamma}$ is the fundamental torsion tensor in our Riemannian space with a built-in tetrad lattice (5).

All the invariants bilinear in the $g^{\hat{\alpha}}_{\alpha;\beta}$ can now be written in the form

$$I = T^{\alpha\beta\gamma}_{\lambda\mu\nu}\,\Lambda_{[\alpha\beta]\gamma}\,\Lambda^{[\lambda\mu]\nu}.$$

The use of the conditions I and II gives us the following seven invariants:

$$I_1 = g^{\hat{:}}\,\Lambda_{[\varrho\sigma]}{}^{\sigma}\,\Lambda^{[\varrho\tau]}{}_{\tau}, \tag{IV.8}$$

$$I_2 = g^{\hat{:}}\,\Lambda_{[\alpha\beta]\gamma}\,\Lambda^{[\gamma\beta]\alpha}, \tag{IV.9}$$

$$I_3 = g^{\hat{:}}\,\Lambda_{[\alpha\beta]\gamma}\,\Lambda^{[\alpha\beta]\gamma}, \tag{IV.10}$$

$$\check{I}_1 = \varepsilon^{\beta\gamma\nu\mu}\,g^{\alpha\lambda}\,\Lambda_{[\alpha\beta]\gamma}\,\Lambda_{[\lambda\nu]\mu}, \tag{IV.11}$$

$$\check{I}_2 = \varepsilon^{\alpha\beta\nu\mu}\,g^{\gamma\lambda}\,\Lambda_{[\alpha\beta]\gamma}\,\Lambda_{[\lambda\nu]\mu}, \tag{IV.12}$$

$$\check{I}_3 = \varepsilon^{\alpha\lambda\nu\mu}\,g^{\beta\gamma}\,\Lambda_{[\alpha\beta]\gamma}\,\Lambda_{[\lambda\nu]\mu}, \tag{IV.13}$$

$$\check{I}_4 = \varepsilon^{\alpha\beta\lambda\nu}\,g^{\gamma\mu}\,\Lambda_{[\alpha\beta]\gamma}\,\Lambda_{[\lambda\nu]\mu}. \tag{IV.14}$$

All these are pseudoscalar densities with respect to the group of coordinate transformations. I_1, I_2, I_3 are pseudoscalar also with respect to constant Lorentz rotations of tetrads, while $\check{I}_1, \cdots, \check{I}_4$ are scalar with respect to this group (see section VI).

The term $\check{I}_4$ can be written as a full divergence; in fact

$$\left.\begin{aligned} \check{I}_4 &= 4\,\varepsilon^{\alpha\beta\lambda\nu}\,g^{\gamma\mu}\,g_{\hat{\varrho}\gamma}\,g_{\hat{\sigma}\mu}\,g^{\hat{\varrho}}_{\alpha,\beta}\,g^{\hat{\sigma}}_{\lambda,\nu} \\ &= 4\left[\varepsilon^{\alpha\beta\lambda\nu}\,g_{\hat{\varrho}\hat{\sigma}}\,g^{\hat{\varrho}}_{\alpha,\beta}\,g^{\hat{\sigma}}_{\lambda}\right]_{,\nu}. \end{aligned}\right\} \tag{IV.15}$$

Using the inverse relation of (IV.6)

$$\Lambda_{[\lambda\nu]\mu} = \frac{i}{2}\,g^{\hat{:}}\,\varepsilon_{\lambda\nu\varrho\sigma}\check{\Lambda}^{[\varrho\sigma]}{}_{\mu} \tag{IV.16}$$

and the relation

$$\Lambda_{[\lambda\nu]\mu} + \Lambda_{[\nu\mu]\lambda} + \Lambda_{[\mu\lambda]\nu} = i g^{\hat{:}}\,\varepsilon_{\varrho\lambda\nu\mu}\,\check{\Lambda}^{[\varrho\sigma]}{}_{\sigma}, \tag{IV.17}$$

we find

$$\left.\begin{aligned} \check{I}_3 &= \varepsilon^{\alpha\lambda\nu\mu}\,g^{\gamma\beta}\,\Lambda_{[\alpha\beta]\gamma}\left(\frac{i}{2}\,g^{\hat{:}}\,\varepsilon_{\varrho\sigma\lambda\nu}\,\check{\Lambda}^{[\varrho\sigma]}{}_{\mu}\right) \\ &= 2\,i\,g^{\gamma\beta}\,\check{\Lambda}^{[\varrho\sigma]}{}_{\sigma}\,\Lambda_{[\varrho\beta]\gamma}, \end{aligned}\right\} \tag{IV.18}$$

$$\left.\begin{aligned}\check{I}_2 &= \frac{1}{2}\,\varepsilon^{\alpha\beta\nu\mu}\,g^{\gamma\lambda}\,\Lambda_{[\alpha\beta]\gamma}\,(\Lambda_{[\lambda\nu]\mu}-\Lambda_{[\lambda\mu]\nu}+\Lambda_{[\nu\mu]\lambda}-\Lambda_{[\nu\mu]\lambda})\\ &= 2\,i\,g^{\lambda\gamma}\,\check{\Lambda}^{[\varrho\sigma]}{}_{\sigma}\,\Lambda_{[\varrho\lambda]\gamma}-\frac{1}{2}\check{I}_4\\ &= \check{I}_3-\frac{1}{2}\check{I}_4,\end{aligned}\right\}\quad \text{(IV.19)}$$

$$\left.\begin{aligned}\check{I}_1 &= \frac{1}{2}\,\varepsilon^{\beta\gamma\nu\mu}\,g^{\alpha\lambda}\,\Lambda_{[\alpha\beta]\gamma}\,(\Lambda_{[\lambda\nu]\mu}-\Lambda_{[\lambda\mu]\nu}+\Lambda_{[\nu\mu]\lambda}-\Lambda_{[\nu\mu]\lambda})\\ &= -\frac{1}{2}\check{I}_2-\frac{1}{2}\check{I}_3\\ &= -\check{I}_3+\frac{1}{4}\check{I}_4.\end{aligned}\right\}\quad \text{(IV.20)}$$

It is clear that of the invariants $\check{I}_1,\cdots,\check{I}_4$ only one among the first three need be considered in the Lagrangian; we select for later use the term $\check{I}_3$.

Let us now turn to the other invariants, I_1, I_2, I_3. It will be useful to introduce some linear combination of them, such as

$$P_1=\frac{1}{2}I_2+\frac{1}{4}I_3-I_1=\hat{g}\,[\gamma_{\alpha\beta\gamma}\,\gamma^{\alpha\gamma\beta}-\Phi_\alpha\,\Phi^\alpha], \quad \text{(IV.21)}$$

$$P_2=I_2-\frac{1}{2}I_3=-\frac{1}{16}\hat{g}\,\check{\Lambda}^\alpha\,\check{\Lambda}_\alpha \quad \text{(IV.22)}$$

$$P_3=\frac{1}{2}I_2+\frac{3}{4}I_3=\hat{g}\,\gamma_{\alpha\beta\gamma}\,\gamma^{\alpha\beta\gamma}. \quad \text{(IV.23)}$$

It is easy to see that P_1 is equal to $\mathfrak{L}_M$, the Møller gravitational Lagrangian, differing from $\sqrt{-g}\,R$ only by a divergence (see appendix A). Our general Lagrangian now takes the form

$$\mathfrak{L}=\sum_1^3{}_i\,a_i\,P_i+a_4\,\check{I}_3, \quad \text{(IV.24)}$$

and in it we have four arbitrary constants.

V. The Linear Approximation and the Complete Determination of the Lagrangian

We shall study in this section some consequences derivable from the Lagrangian (IV.24) in order to get information on the constants $a_1,\cdots,a_4$. The approach used is the study of the linear form of the theory; in particular we require that the linearized field equations are the same as the Einstein ones plus, of course, a set of six equations. To put it in a different form, we ask that our theory contains the Newtonian theory of gravitation as a limiting case.

To perform the linearization we consider an "insular" system of matter and assume that space-time is asymptotically flat and that cartesian coordinates are used at infinity. Then the tetrad can be written in the form

$$g^{\hat{\varrho}}_{\varrho} = \delta^{\hat{\varrho}}_{\varrho} + \frac{1}{2} h^{\hat{\varrho}}_{\varrho}. \tag{V.1}$$

The term $h^{\hat{\varrho}}_{\varrho}$, describing the deviation of space-time from flatness, is assumed to be everywhere small of the first order. In all the following calculations terms of orders higher than the first in $h^{\hat{\varrho}}_{\varrho}$ will be neglected. Introducing the quantity

$$h_{\alpha\beta} = g_{\hat{\alpha}\hat{\beta}}\, \delta^{\hat{\alpha}}_{\alpha}\, h^{\hat{\beta}}_{\beta} \tag{V.2}$$

and substituting (V.1), (V.2) in (II.5), we find that

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{<\alpha\beta>}, \tag{V.3}$$

where $\eta_{\alpha\beta}$ is the special-relativity metric tensor.

Hence, in the linear approximation, the tetrad field is described by the tensor $h_{\alpha\beta}$, whose symmetric part gives the metric tensor while the antisymmetric part describes the new degrees of freedom of the theory.

The torsion tensor obtained from (V.1) is

$$\Lambda_{[\alpha\beta]\gamma} = \frac{1}{2}[h_{\gamma\alpha,\beta} - h_{\gamma\beta,\alpha}]. \tag{V.4}$$

Substituting (V.4) in (IV.13), (IV.21), (IV.22), (IV.23) and introducing the notation

$$h = h_{<\alpha}{}^{\alpha}{}_{>} \tag{V.5}$$

and the dual

$$\check{h}^{[\alpha\beta]} = -\frac{i}{2}\varepsilon^{\varrho\sigma\alpha\beta}\, h_{[\varrho\sigma]}, \tag{V.6}$$

we obtain

$$\left.\begin{aligned} \mathfrak{L}_M = P_1 = \frac{1}{4} h_{<\alpha\beta>,\gamma}\, h^{<\alpha\beta>,\gamma} - \frac{1}{2} h_{<\alpha\beta>}{}^{,\alpha}\, h^{<\gamma\beta>}{}_{,\gamma} \\ + \frac{1}{2} h_{,\alpha}\, h^{<\alpha\beta>}{}_{,\beta} - \frac{1}{4} h_{,\alpha}\, h^{,\alpha}, \end{aligned}\right\} \tag{V.7}$$

$$P_2 = \check{h}^{[\alpha\beta]}{}_{,\beta}\, \check{h}_{[\alpha\gamma]}{}^{,\gamma}, \tag{V.8}$$

$$\left.\begin{aligned} P_3 = \frac{1}{2} h_{<\alpha\beta>,\gamma}\, h^{<\alpha\beta>,\gamma} - \frac{1}{2} h_{<\alpha\beta>}{}^{,\alpha}\, h^{<\gamma\beta>}{}_{,\gamma} \\ + \frac{1}{4} h_{[\alpha\beta],\gamma}\, h^{[\alpha\beta],\gamma} + h_{<\alpha\beta>}{}^{,\alpha}\, h^{[\gamma\beta]}{}_{,\gamma}, \end{aligned}\right\} \tag{V.9}$$

$$\check{I}_3 = 4\, i\, \check{h}^{[\alpha\beta]}{}_{,\beta} \left[h_{<\alpha\gamma>}{}^{,\gamma} - h_{[\alpha\gamma]}{}^{,\gamma} \right]. \tag{V.10}$$

In writing down (V.7), . . . , (V.10), we have neglected divergences; in particular, the expansion of $\mathfrak{L}_M$ has been replaced by that of $\sqrt{-g}\, R$ since, of course, they can only differ by a divergence.

The equations obtained by varying $\mathfrak{L}_M$ with respect to $h_{<\alpha\beta>}$ are just the Einstein weak-field equations. This means that the constants a_2, a_3, a_4 must be so chosen that in the equation obtained by varying the Lagrangian with respect to $h_{<\alpha\beta>}$ all terms depending on $h_{<\alpha\beta>}$ itself come only from $\mathfrak{L}_M$. The first thing one would think of is then to assume

$$a_3 = a_4 = 0$$

so that the Langrangian becomes

$$\mathfrak{L} = a_1\, \mathfrak{L}_M + a_2\, P_2 .$$

The linearized field equations, in the absence of matter, are then

$$\frac{1}{2}\left\{ - \Box\, \varphi_{\alpha\beta} - \eta_{\alpha\beta}\, \varphi^{\varrho\sigma}{}_{,\varrho,\sigma} + (\eta_{\alpha\varrho}\, \delta^{\gamma}_{\beta} + \eta_{\beta\varrho}\, \delta^{\gamma}_{\alpha})\, \varphi^{\varrho\sigma}{}_{,\sigma,\gamma} \right\} = 0 , \tag{V.11}$$

$$\check{h}^{[\alpha\beta]}{}_{,\beta}{}^{,\gamma} - \check{h}^{[\gamma\beta]}{}_{,\beta}{}^{,\alpha} = 0 , \tag{V.12}$$

$$\varphi_{\alpha\beta} = h_{<\alpha\beta>} - \frac{1}{2}\, \eta_{\alpha\beta}\, h . \tag{V.13}$$

Equations (V.11), (V.12) cannot be solved unless we specify the boundary conditions. To derive the linear approximation we have assumed an asymptotically flat space-time in which cartesian coordinates are used at infinity. Hence we can introduce the "outgoing waves" boundary conditions (5)

(a) $$\lim_{r \to \infty} g^{\hat{\alpha}}_{\alpha} = \delta^{\hat{\alpha}}_{\alpha};$$

(b) if ψ is any of the quantities $g^{\alpha}_{\hat{\alpha}} - \delta^{\alpha}_{\hat{\alpha}}$ or $g_{\alpha\beta} - \eta_{\alpha\beta}$, it must satisfy the condition

$$\lim_{r \to \infty} \left\{ \frac{\partial (r\,\psi)}{\partial r} + \frac{1}{c} \frac{\partial (r\,\psi)}{\partial t} \right\} = 0$$

for all values of $t_0 = t + \frac{r}{c}$ in an arbitrary fixed interval; the ψ and the first-order derivatives must also be bounded everywhere and must go to zero at least like $1/r$ for $r \to \infty$.

The conditions a, b require the matter system to be an insular system, but they do not exclude the presence of gravitational waves emitted by it. An important consequence [(5)] of a, b is that, given a quantity ψ satisfying b, the only solution of the equation

$$\Box\, \psi = 0 \tag{V.14}$$

is

$$\psi = 0. \tag{V.15}$$

This result allows us to solve equation (V.12); in fact we obtain from it by derivation

$$\Box\, \check{h}^{[\alpha\beta]}{}_{,\beta} = 0$$

and, using (V.14), (V.15),

$$\check{h}^{[\alpha\beta]}{}_{,\beta} = 0. \tag{V.16}$$

Equation (V.16) just says that $h_{[\mu\nu]}$ can be written by means of a vector potential A_μ as

$$h_{[\mu\nu]} = A_{\mu,\nu} - A_{\nu,\mu}. \tag{V.17}$$

For an arbitrary vector field A_μ the $h_{[\mu\nu]}$ given by (V.17) satisfy the field equations (V.12); hence these cannot determine the skew degrees of freedom of the tetrad field.

In this situation it is impossible to attribute any physical meaning to the $h_{[\mu\nu]}$. The Lagrangian we obtain when choosing $a_3 = a_4 = 0$ is thus completely unsatisfactory, although we can derive the linearized Einstein equations from it.

A choice of constants giving a better result is

$$a_2 = a_3 = 0 \qquad a_1 \neq 0 \qquad a_4 \neq 0.$$

In this case the field equations are

$$\left.\begin{aligned} &\frac{a_1}{2}\left\{-\Box\, \varphi^{\mu\nu} - \eta^{\mu\nu}\,\varphi^{\alpha\beta}{}_{,\alpha,\beta} + (\delta^{\mu}_{\alpha}\,\eta^{\gamma\nu} + \delta^{\nu}_{\alpha}\,\eta^{\gamma\mu})\,\varphi^{\alpha\beta}{}_{,\beta,\gamma}\right\} \\ &\qquad - 2\, i a_4 \left\{\check{h}^{[\nu\beta]}{}_{,\beta},{}^{\mu} + \check{h}^{[\mu\beta]}{}_{,\beta},{}^{\nu}\right\} = -\, T^{<\mu\nu>}, \end{aligned}\right\} \tag{V.18}$$

$$-2\,\varepsilon^{\mu\nu\alpha\beta}\left\{h_{<\alpha\gamma>,\beta},{}^{\gamma} - h_{[\alpha\gamma],\beta},{}^{\gamma}\right\} + 2\, i\left\{\check{h}^{[\mu\beta]}{}_{,\beta},{}^{\nu} - \check{h}^{[\nu\beta]}{}_{,\beta},{}^{\mu}\right\} = 0. \tag{V.19}$$

To derive (V.18) we added to our Lagrangian a matter term $\mathfrak{L}_m$ depending on the $g^{\alpha}_{\dot\alpha}$ only through the metric tensor, so that

$$\frac{\delta\, \mathfrak{L}_m}{\delta\, h_{<\mu\nu>}} = T^{<\mu\nu>}$$

$$\frac{\delta\, \mathfrak{L}_m}{\delta\, h_{[\mu\nu]}} = 0.$$

All the macroscopic physical systems are of this type, so that equations (V.18), (V.19) are what we need in order to see whether the linear form of our theory contains the Newtonian theory of gravitation.

Taking the derivative with respect to x^ν of (V.19), we obtain

$$\Box \check{h}^{[\mu\beta]}{}_{,\beta} = 0 \tag{V.20}$$

and, using (V.14), (V.15),

$$\check{h}^{[\mu\beta]}{}_{,\beta} = 0. \tag{V.21}$$

This result reduces (V.18) to the Einstein weak-field equations. The other equation, (V.19), still contains both the symmetric and the skew part of $h_{\mu\nu}$.

A complete separation can be achieved in harmonic coordinates, where the de Donder condition

$$\varphi^{\alpha\beta}{}_{,\beta} = 0 \tag{V.22}$$

holds. This, together with (V.21), allows us to write (V.18), (V.19) as

$$-a_1 \Box \varphi^{\mu\nu} = 2\, T^{<\mu\nu>}, \tag{V.23}$$

$$h_{[\alpha\beta],\beta}{}^{,\gamma} - h_{[\beta\gamma],\alpha}{}^{,\gamma} = 0. \tag{V.24}$$

Since from (V.21) it follows again that

$$h^{[\alpha\beta]} = A_{\alpha,\beta} - A_{\beta,\alpha},$$

(V.24) becomes simply

$$\Box\, h_{[\alpha\beta]} = 0 \tag{V.25}$$

so that eventually our field equations are reduced to

$$-a_1 \Box \varphi^{\mu\nu} = 2\, T^{<\mu\nu>}, \tag{V.23}$$

$$h_{[\mu\nu]} = 0. \tag{V.26}$$

This is the same result as that obtained by Møller[(5)]. From (V.23) we can determine the value of the constant a_1:

$$a_1^{-1} = \frac{8\pi k}{c^4}, \tag{V.27}$$

where k is the Newtonian gravitational constant. These results allow us to assume as Lagrangian of the tetrad field

$$\mathfrak{L} = k_1 \mathfrak{L}_M + k_2 \check{I}_3 \tag{V.28}$$

with $k_1 = a_1$ given by (V.27).

About the other constant, k_2, we have no information; since k_2 drops out of the equations also in the case of a spherically symmetric system, for which the solution of the field equations derivable from (V.28) will be given in section VIII, we shall not be able to give its value in this paper. We have a feeling that k_2 is related to some non-classical aspect of the gravitational field and that it might become important in a quantized version of this theory.

We would also point out that the results (V.23), (V.26), and also those obtained in section VIII for the Schwarzschild case, are left unchanged if we add to the Lagrangian (V.28) the term $a_2 P_2$. For simplicity we assume $a_2 = 0$, but the possibility of adding the term $a_2 P_2$ to our Lagrangian is worth noting.

VI. The Invariance Properties of the Lagrangian with Respect to the Group of Tetrad Rotations

In this section we want to study the behaviour of the Lagrangian

$$\mathfrak{L} = k_1 \mathfrak{L}_M + k_2 \check{I}_3 \tag{VI.1}$$

with respect to the group of tetrad rotations, defined by

$$g'^{\hat{\alpha}}_{\alpha}(x) = L^{\hat{\alpha}}{}_{\hat{\beta}}(x)\, g^{\hat{\beta}}_{\alpha}(x), \tag{VI.2}$$

$$g_{\hat{\varrho}\hat{\sigma}} L^{\hat{\varrho}}{}_{\hat{\alpha}}(x) L^{\hat{\sigma}}{}_{\hat{\beta}}(x) = g_{\hat{\alpha}\hat{\beta}} \tag{VI.3}$$

and already introduced in section II.

The matrix $L^{\hat{\alpha}}{}_{\hat{\beta}}$ can in general be a function of the coordinates. We have already noted that the metric tensor is invariant under the substitution (VI.2). The same is true of the Ricci tensor $R^{\alpha\beta}$ and of the scalar curvature R, in accordance with the fact that the Einstein field equations cannot alone fix the tetrad field.

Let us now consider the Lagrangian (VI.1). It is clear that under constant tetrad rotations, i. e. when $L^{\hat{\alpha}}{}_{\hat{\beta},\mu} = 0$, all the tensors not containing tetrad indices, like $\Lambda_{[\alpha\beta]\gamma}$, will be transformed like scalars. On the other hand, the quantity $g\hat{:}$ is transformed like a density:

$$g'\hat{:} = g\hat{:} \quad \det \left| L^{\hat{\alpha}}{}_{\hat{\beta}} \right|. \tag{VI.4}$$

It follows that the two terms $\mathfrak{L}_M$ and $\check{I}_3$ defined by (IV.21) and (IV.13) are transformed like a pseudoscalar and a scalar density respectively under the whole rotation group. The Lagrangian $\mathfrak{L}$ is invariant only with respect to the subgroup of proper tetrad rotations.

Let us now look into the general case $L^{\hat{\alpha}}{}_{\hat{\beta},\mu} \neq 0$, but limiting ourselves to the subgroup of proper rotations. For simplicity, the infinitesimal transformation

$$L^{\hat{\alpha}}{}_{\hat{\beta}} = \delta^{\hat{\alpha}}{}_{\hat{\beta}} + \varepsilon^{\hat{\alpha}}{}_{\hat{\beta}}, \tag{VI.5}$$

for which the condition (VI.3) becomes

$$\varepsilon_{\hat{\alpha}\hat{\beta}} = -\varepsilon_{\hat{\beta}\hat{\alpha}}, \tag{VI.6}$$

will be considered.

After a somewhat long but simple calculation the variation of $\mathfrak{L}$ can be written as

$$\delta\,\mathfrak{L} = 2\,k_1\,g^{\hat{}}_{\cdot}\,(W^{\mu}_{\hat{\alpha}\hat{\beta}}\,\varepsilon^{\hat{\alpha}\hat{\beta}})_{;\,\mu} + k_2\,Z^{\mu}_{\hat{\alpha}\hat{\beta}}\,\varepsilon^{\hat{\alpha}\hat{\beta}}{}_{,\,\mu}, \tag{VI.7}$$

where

$$W^{\mu}_{\hat{\alpha}\hat{\beta}} = (g^{\mu}_{\hat{\alpha};\,\lambda} - g^{\nu}_{\hat{\alpha};\,\nu}\,\delta^{\mu}_{\lambda})\,g^{\lambda}_{\hat{\beta}} - (g^{\mu}_{\hat{\beta};\,\lambda} - g^{\nu}_{\hat{\beta};\,\nu}\,\delta^{\mu}_{\lambda})\,g^{\lambda}_{\hat{\alpha}} \tag{VI.8}$$

and

$$\left.\begin{aligned} Z^{\mu}_{\hat{\alpha}\hat{\beta}} = \varepsilon^{\alpha\beta\gamma\delta}\,g^{\lambda\nu}\{&\Lambda_{[\alpha\lambda]\,\nu}\,g_{\hat{\beta}\delta}\,(\delta^{\varrho}_{\beta}\,\delta^{\mu}_{\gamma} - \delta^{\varrho}_{\gamma}\,\delta^{\mu}_{\beta}) \\ &+ \Lambda_{[\beta\gamma]\,\delta}\,g_{\hat{\beta}\lambda}\,(\delta^{\varrho}_{\alpha}\,\delta^{\mu}_{\nu} - \delta^{\varrho}_{\nu}\,\delta^{\mu}_{\alpha})\}\,g_{\hat{\alpha}\varrho}. \end{aligned}\right\} \tag{VI.9}$$

Note that the variation of $\mathfrak{L}_M$ is a four-divergence, as it should be, since the theory deduced only from this term is equivalent to the Einstein theory.

The quantity $W^{\mu}_{\hat{\alpha}\hat{\beta}}$ has the property that

$$W^{\mu}_{\hat{\alpha}\hat{\beta};\,\mu} = 0. \tag{VI.10}$$

This can be seen more easily if we use absolute derivatives and the identity (5)

$$\Lambda_{[\alpha\beta]}{}^{\mu}{}_{/\mu} + \Phi_{\beta/\alpha} - \Phi_{\alpha/\beta} - \Lambda_{[\alpha\beta]\,\mu}\,\Phi^{\mu} = 0. \tag{VI.11}$$

A stroke, $A_{\alpha/\beta}$, here means absolute derivative. In fact, from the connection between the absolute and the covariant derivative (5) we obtain

$$\left.\begin{aligned} W^{\mu}_{\hat{\alpha}\hat{\beta};\,\mu} &= W_{\hat{\alpha}\hat{\beta}}{}^{\mu}{}_{/\mu} - W_{\hat{\alpha}\hat{\beta}}{}^{\mu}\,\Phi_{\mu} \\ &= g^{\alpha}_{\hat{\alpha}}\,g^{\beta}_{\hat{\beta}}\,(W_{\alpha\beta}{}^{\mu}{}_{/\mu} - W_{\alpha\beta}{}^{\mu}\,\Phi_{\mu}). \end{aligned}\right\} \tag{VI.12}$$

From the definition (VI.8) of $W^{\mu}_{\hat{\alpha}\hat{\beta}}$ we find that

$$W_{\alpha\beta}{}^{\mu} = -\Lambda_{[\alpha\beta]}{}^{\mu} + \delta^{\mu}_{\beta}\,\Phi_{\alpha} - \delta^{\mu}_{\alpha}\,\Phi_{\beta}. \tag{VI.13}$$

Substituting (VI.13) in (VI.12) and using the identity (VI.11), we find the result (VI.10).

The variation of the Lagrangian can now be written simply as

$$\delta\,\mathfrak{L} = (2\,k_1\,g^{\hat{}}_{\cdot}\,W^{\mu}_{\hat{\alpha}\hat{\beta}} + k_2\,Z^{\mu}_{\hat{\alpha}\hat{\beta}})\,\varepsilon^{\hat{\alpha}\hat{\beta}}{}_{,\,\mu}. \tag{VI.14}$$

We see that $\mathfrak{L}$, contrary to the Einstein Lagrangian, is not invariant under a position-dependent tetrad rotation. The condition

$$\delta\,\mathfrak{L} \equiv 0 \tag{VI.15}$$

can be satisfied only when we assume

$$\varepsilon_{\hat{\alpha}\hat{\beta},\,\mu} = 0. \tag{VI.16}$$

It follows that the Lagrangian (VI.1) is invariant only with respect to the group of proper constant tetrad rotations. The existence of this "gauge" group will be used in appendix B to introduce spinors.

VII. The Field Equations

In this section we want to derive explicitly the field equations from our Lagrangian, which we now write as

$$\mathfrak{L} = \frac{1}{2} k_1 \mathfrak{L}_M + k_2 \check{I}_3 + \frac{1}{2} \mathfrak{L}_m^{(b)} + \mathfrak{L}_m^{(f)}. \tag{VII.1}$$

The Lagrangian of matter has been divided into two parts; the first, $\mathfrak{L}_m^{(b)}$, depends on the tetrads only through the metric tensor, whereas this assumption does not apply to the second, $\mathfrak{L}_m^{(f)}$.

Examples of physical systems of the first kind are all the classical systems, such as the electromagnetic field, a hydrodynamical system and a field of boson particles. To the second type belongs the Lagrangian of fermions, like electrons and neutrinos, which must be written in terms of spinors.

To evalute the variation of the action integral we first consider the terms $\mathfrak{L}_M$ and $\mathfrak{L}_m^{(b)}$ of the Lagrangian. Since

$$\delta \int_\Omega (k_1 \mathfrak{L}_M + \mathfrak{L}_m^{(b)})\, d^4 x = \delta \int_\Omega (k_1 \mathfrak{R} + \mathfrak{L}_m^{(b)})\, d^4 x, \tag{VII.2}$$

we have

$$\left.\begin{aligned} &\int_\Omega \frac{\delta}{\delta g^{\hat{\varrho}}_{\varrho}} \{k_1 \mathfrak{L}_M + \mathfrak{L}_m^{(b)}\}\, \delta g^{\hat{\varrho}}_{\varrho}\, d^4 x \\ &= \int_\Omega \frac{\delta}{\delta g_{\varrho\sigma}} \{k_1 \mathfrak{R} + \mathfrak{L}_m^{(b)}\}\, \delta g_{\varrho\sigma}\, d^4 x. \end{aligned}\right\} \tag{VII.3}$$

But

$$\delta g_{\alpha\beta} = g_{\hat{\varrho}\beta}\, \delta g^{\hat{\varrho}}_{\alpha} + g_{\hat{\varrho}\alpha}\, \delta g^{\hat{\varrho}}_{\beta} \tag{VII.4}$$

so that we obtain

$$\delta \int_\Omega (k_1 \mathfrak{L}_M + \mathfrak{L}_m^{(b)})\, d^4 x = 2 \int_\Omega \overset{\frown}{g} \colon (k_1 G^{\alpha\beta} + T^{(b)\,\alpha\beta})\, g_{\hat{\varrho}\beta}\, \delta g^{\hat{\varrho}}_{\alpha}\, d^4 x, \tag{VII.5}$$

where

$$\left.\begin{aligned} G^{\alpha\beta} &= R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R, \\ g^{\hat{:}}\, T^{(b)\,\alpha\beta} &= \frac{\delta\, \mathfrak{L}_m^{(b)}}{\delta g_{\alpha\beta}}, \end{aligned}\right\} \quad \text{(VII.6)}$$

and $R^{\alpha\beta}$ is the Ricci tensor.

On introduction of the quantities

$$g^{\hat{:}}\, F_{\hat{\varrho}}^{\ \varrho} = \frac{\delta \check{I}_3}{\delta g_{\varrho}^{\ \hat{\varrho}}} \quad \text{(VII.7)}$$

$$g^{\hat{:}}\, T^{(f)}{}_{\hat{\varrho}}^{\ \varrho} = \frac{\delta \mathfrak{L}_m^{(f)}}{\delta g_{\varrho}^{\ \hat{\varrho}}} \quad \text{(VII.8)}$$

the variational principle can be written as

$$\left.\begin{aligned} 0 &= \delta \int_{\Omega} \left(\frac{1}{2} k_1 \mathfrak{L}_M + k_2 \check{I}_3 + \frac{1}{2} \mathfrak{L}_m^{(b)} + \mathfrak{L}_m^{(f)} \right) d^4 x \\ &= \int_{\Omega} g^{\hat{:}} \left(k_1 G_{\hat{\varrho}}^{\ \varrho} + k_2 F_{\hat{\varrho}}^{\ \varrho} + T^{(b)}{}_{\hat{\varrho}}^{\ \varrho} + T^{(f)}{}_{\hat{\varrho}}^{\ \varrho} \right) \delta g_{\varrho}^{\hat{\varrho}}\, d^4 x, \end{aligned}\right\} \quad \text{(VII.9)}$$

and the resulting field equations are

$$k_1 G_{\hat{\varrho}}^{\ \varrho} + k_2 F_{\hat{\varrho}}^{\ \varrho} + T^{(b)}{}_{\hat{\varrho}}^{\ \varrho} + T^{(f)}{}_{\hat{\varrho}}^{\ \varrho} = 0. \quad \text{(VII.10)}$$

The evaluation of $F_{\hat{\varrho}}^{\ \varrho}$ and of $F^{\alpha\beta} = g^{\hat{\varrho}\alpha} F_{\hat{\varrho}}^{\ \beta}$ is performed in appendix A.

Multiplying (VII.10) by $g^{\hat{\varrho}\alpha}$ and using (A.24), (A.25), we obtain another form of the field equations:

$$\left.\begin{aligned} k_1 G^{\alpha\beta} + k_2 \Big\{ & \check{\Lambda}^{\beta;\,\alpha} + 2\, \eta^{\beta\alpha\varrho\sigma} \Phi_{\varrho;\,\sigma} - \Phi^{\alpha} \check{\Lambda}^{\beta} \\ & + \check{\Lambda}^{\gamma} \gamma^{\beta\alpha}{}_{\gamma} + 2\, \eta^{\beta\varrho\sigma\tau} \Phi_{\varrho} \Lambda_{[\sigma\tau]}{}^{\alpha} + \check{\Lambda}^{\varrho} \Lambda_{[\varrho\cdot]}{}^{\alpha\beta} \\ & + \frac{1}{2} g^{\alpha\beta} \eta^{\varrho\sigma\tau\varsigma} \Lambda_{[\tau\varsigma]\,\omega} \Lambda_{[\varrho\sigma]}{}^{\omega} \Big\} \\ & = - T^{(b)\,\alpha\beta} - T^{(f)\,\alpha\beta}. \end{aligned}\right\} \quad \text{(VII.11)}$$

While $G^{\alpha\beta}$ and $T^{(b)\,\alpha\beta}$ are symmetric tensors, $F^{\alpha\beta}$ and $T^{(f)\,\alpha\beta}$ have no well-defined symmetry property. Both $T^{(b)\,\alpha\beta}$ and $T^{(f)\,\alpha\beta}$ differ from the canonical energy-momentum matter tensor by a four-divergence.

Since our theory is generally covariant, the field equations must contain a set of four identities, in the same way as the Einstein equations are supplemented by the

Bianchi identities. To obtain these identities we apply the method of infinitesimal coordinate transformations to the scalar density $\mathfrak{L}$*.

Therefore, under an arbitrary infinitesimal coordinate transformation

$$x'^{\alpha} = x^{\alpha} + \xi^{\alpha}(x) \tag{VII.12}$$

the local variation of $\mathfrak{L}$ is given by

$$\delta \mathfrak{L} = -(\mathfrak{L}\xi^{\alpha})_{,\alpha}. \tag{VII.13}$$

Integrating (VII.13) over a finite region Ω in space-time, we obtain, for all functions ξ^{α} which vanish on the boundary of Ω together with their first-order derivatives,

$$\int_{\Omega} \delta \mathfrak{L}\, d^4 x = \int_{\Omega} \frac{\delta \mathfrak{L}}{\delta g_{\hat{\alpha}\alpha}} \delta g_{\hat{\alpha}\alpha}\, d^4 x = 0, \tag{VII.14}$$

where

$$\frac{\delta \mathfrak{L}}{\delta g_{\hat{\alpha}\alpha}} = \frac{\partial \mathfrak{L}}{\partial g_{\hat{\alpha}\alpha}} - \left(\frac{\partial \mathfrak{L}}{\partial g_{\hat{\alpha}\alpha,\beta}}\right)_{,\beta}.$$

Substituting

$$\delta g_{\hat{\alpha}\alpha} = -g_{\hat{\alpha}\beta}\,\xi^{\beta}{}_{,\alpha} - g_{\hat{\alpha}\alpha,\beta}\,\xi^{\beta} \tag{VII.15}$$

in (VII.14), we obtain, after a partial integration,

$$\int_{\Omega} \left[\left(\frac{\delta \mathfrak{L}}{\delta g_{\hat{\alpha}\alpha}} g_{\hat{\alpha}\beta}\right)_{,\alpha} - \frac{\delta \mathfrak{L}}{\delta g_{\hat{\alpha}\alpha}} g_{\hat{\alpha}\alpha,\beta}\right] \xi^{\beta}\, d^4 x = 0. \tag{VII.16}$$

As the functions ξ^{β} are arbitrary inside Ω, the identities

$$\left(\frac{\delta \mathfrak{L}}{\delta g_{\hat{\alpha}\alpha}} g_{\hat{\alpha}\beta}\right)_{,\alpha} - \frac{\delta \mathfrak{L}}{\delta g_{\hat{\alpha}\alpha}} g_{\hat{\alpha}\alpha,\beta} = 0 \tag{VII.17}$$

must hold.

Since in our case $\mathfrak{L} = \frac{1}{2} k_1 \mathfrak{L}_M + k_2 \check{I}_3$, (VII.17) can be written as

$$[g^{\hat{:}} (k_1 G^{\hat{\alpha}\alpha} + k_2 F^{\hat{\alpha}\alpha})\, g_{\hat{\alpha}\beta}]_{,\alpha} - g^{\hat{:}} (k_1 G^{\hat{\alpha}\alpha} + k_2 F^{\hat{\alpha}\alpha})\, g_{\hat{\alpha}\alpha,\beta} = 0$$

or, on introduction of the quantity

$$\Delta^{\varrho}{}_{\alpha\beta} = g^{\varrho}_{\hat{\alpha}}\, g^{\hat{\alpha}}{}_{\alpha,\beta}, \tag{VII.18}$$

as

$$[g^{\hat{:}} (k_1 G_{\beta}{}^{\alpha} + k_2 F_{\beta}{}^{\alpha})]_{,\alpha} - g^{\hat{:}} (k_1 G_{\varrho}{}^{\alpha} + k_2 F_{\varrho}{}^{\alpha})\, \Delta^{\varrho}{}_{\alpha\beta} = 0. \tag{VII.19}$$

* See, for instance, reference 10 and also C. Møller, Proceedings of the Warsaw Conference on General Relativity.

Using the relation[5]

$$\Delta^{\alpha}{}_{\beta\gamma} = \Gamma^{\alpha}{}_{\beta\gamma} + \gamma^{\alpha}{}_{\beta\gamma}, \tag{VII.20}$$

we have from (VII.19)

$$(k_1 G_{\beta}{}^{\alpha} + k_2 F_{\beta}{}^{\alpha})_{;\alpha} - (k_1 G_{\varrho}{}^{\alpha} + k_2 F_{\varrho}{}^{\alpha}) \gamma^{\varrho}{}_{\alpha\beta} = 0,$$

or, taking into account the symmetry properties of $G^{\alpha\beta}$ and $\gamma_{\varrho\alpha\beta}$,

$$k_1 G^{\alpha\beta}{}_{;\alpha} + k_2 [F^{\beta\alpha}{}_{;\alpha} - F^{\varrho\alpha} \gamma_{\varrho\alpha}{}^{\beta}] = 0. \tag{VII.21}$$

In case $k_2 = 0$, (VII.21) becomes the usual Bianchi identities.

An analogous result can be derived for the matter tensor, i. e.

$$T^{(b)\alpha\beta}{}_{;\beta} = 0 \tag{VII.22}$$

$$T^{(f)\alpha\beta}{}_{;\beta} - T^{(f)\varrho\beta} \gamma_{\varrho\beta}{}^{\alpha} = 0. \tag{VII.23}$$

That this is true can be seen by applying the method used to derive (VII.21) to the scalar densities $\mathfrak{L}_m^{(b)}$ and $\mathfrak{L}_m^{(f)}$ and keeping in mind that the variation of $\mathfrak{L}_m$ with respect to the matter field variables vanishes when the field equations are satisfied.

Going back to the field equations, we note that it is possible to separate them into two independent sets, viz

$$\left.\begin{cases} k_1 G^{\alpha\beta} + k_2 F^{<\alpha\beta>} = -T^{(b)\alpha\beta} - T^{(f)<\alpha\beta>} \\ k_2 F^{[\alpha\beta]} = -T^{(f)[\alpha\beta]}. \end{cases}\right\} \tag{VII.24}$$

We saw already in the linearized case the usefulness of this separation.

In general (VII.24) shows that, contrary to what happens in the standard Einstein theory, the space-time structure is determined by matter not only through the symmetric energy-momentum tensors $T^{(b)\alpha\beta}$ and $T^{(f)<\alpha\beta>}$ but also through the skew term $T^{(f)[\alpha\beta]}$.

To acquire an insight into the meaning of this fact, let us consider explicitly a system formed by a Dirac particle with its own gravitational field. The Lagrangian of this system can be written as

$$\mathfrak{L} = \frac{1}{2} k_1 \mathfrak{L}_M + k_2 \check{I}_3 + \mathfrak{L}_D; \tag{VII.25}$$

$\mathfrak{L}_D$ is assumed to be given by

$$\mathfrak{L}_D = \frac{i}{2} g^{\hat{:}} \{\psi^+ (\alpha^{\mu} \psi_{,\mu} - im\, \beta\psi) - (\psi^+_{,\mu} \alpha^{\mu} + im \psi^+ \beta) \psi\} \tag{VII.26}$$

(see appendix B).

Note that in our formalism $\psi_{,\mu}$ is simply the usual derivative $\dfrac{\partial \psi}{\partial x^{\mu}}$.

Performing the variation of $\mathfrak{L}_D$ with respect to ψ^+ and ψ, we arrive at the field equations

$$\left.\begin{cases} \alpha^\mu \psi_{,\mu} - im\,\beta\psi - \frac{1}{2}\alpha^\mu \Phi_\mu \psi = 0 \\ \psi^+{}_{,\mu}\alpha^\mu + im\,\psi^+\beta - \frac{1}{2}\psi^+\alpha^\mu \Phi_\mu = 0. \end{cases}\right\} \quad \text{(VII.27)}$$

To obtain (VII.27) the identity

$$g_{\hat{\alpha}}{}^{\mu}{}_{;\mu} = g_{\hat{\alpha}}{}^{\varrho}\, g_{\hat{\beta}\varrho}\, g^{\hat{\beta}\mu}{}_{;\mu} = -g_{\hat{\alpha}}{}^{\varrho}\,\Phi_\varrho \quad \text{(VII.28)}$$

has been used.

From the form (VII.26) of $\mathfrak{L}_D$, the continuity equation

$$J^\mu{}_{;\mu} \equiv (\psi^+\alpha^\mu\psi)_{;\mu} = 0 \quad \text{(VII.29)}$$

is easily seen to hold.

Equations (VII.27) are reduced to the usual Dirac equations in the case of a flat space, since here we have $\Lambda_{[\alpha\beta]\gamma} = 0$, $\Phi_\alpha = 0$. In a space with torsion the extra term $-\frac{1}{2}\alpha^\mu \Phi_\mu \psi$ represents the interaction between the fermion and gravitation. The similarity of this term to the one introduced in the same equations by the coupling with the electromagnetic field is worth noting.

The tensor $T^{(f)\,\alpha\beta}$ appearing in (VII.11) can easily be derived; in fact

$$\left.\begin{aligned} g^{\hat{:}}\, T_D{}^{\alpha\beta} &= g_{\hat{\alpha}}{}^{\alpha}\frac{\delta\mathfrak{L}_D}{\delta g_{\hat{\alpha}\beta}} \\ &= g^{\alpha\beta}\,\mathfrak{L}_D - \frac{i}{2} g^{\hat{:}}\left\{\psi^+\alpha^\alpha\psi^{,\beta} - \psi^{+,\beta}\alpha^\alpha\psi\right\}. \end{aligned}\right\} \quad \text{(VII.30)}$$

As a consequence of the field equations (VII.27), $\mathfrak{L}_D$ vanishes so that only the second term of (VII.30) has to be taken into account.

We now consider the weak-field limit of the equation

$$k_1 G^{\alpha\beta} + k_2 F^{\alpha\beta} = -T_D{}^{\alpha\beta}. \quad \text{(VII.31)}$$

In harmonic coordinates, i. e. when we use the coordinate condition (V.22), this is

$$-\frac{1}{2}k_1 \Box\,\varphi^{\alpha\beta} + k_2\left\{\varepsilon^{\alpha\gamma\mu\nu} h_{[\nu\mu],\gamma}{}^{,\beta} - \varepsilon^{\alpha\beta\varrho\sigma} h_{[\tau\varrho],\sigma}{}^{,\tau}\right\} = -T_{D0}^{\alpha\beta}, \quad \text{(VII.32)}$$

where

$$T_{D0}^{\alpha\beta} = \frac{i}{2}\left\{\psi^+\tilde{\alpha}^\alpha\psi^{,\beta} - \psi^{+,\beta}\tilde{\alpha}^\alpha\psi\right\} \quad \text{(VII.33)}$$

and

$$\tilde{\alpha}^\alpha = \delta^\alpha_{\hat{\beta}}\,\alpha^{\hat{\beta}}. \quad \text{(VII.34)}$$

The quantity $T_{D0}^{\alpha\beta}$ is the special-relativity energy-momentum tensor of a Dirac particle. It satisfies the relation

$$T_{D0}^{\alpha\beta}{}_{,\beta} = 0, \qquad \text{(VII.35)}$$

as can easily be verified by means of the zero-order Dirac equations

$$\left.\begin{cases} \tilde{a}^{\mu}\,\psi_{,\mu} - im\,\beta\psi = 0 \\ \psi^{+}{}_{,\mu}\,\tilde{a}^{\mu} + im\,\psi^{+}\beta = 0. \end{cases}\right\} \qquad \text{(VII.36)}$$

Taking the derivative with respect to x^{β} of (VII.32) and using (VII.35), we find

$$\Box\,\varepsilon^{\alpha\lambda\nu\mu}\,h_{[\mu\nu],\lambda} = -\frac{i}{2}\Box\,\check{h}^{[\alpha\lambda]}{}_{,\lambda} = 0. \qquad \text{(VII.37)}$$

If again we supplement equations (VII.32) with the boundary conditions a, b used in section V, it follows from (VII.37) that

$$\check{h}^{[\alpha\lambda]}{}_{,\lambda} = 0, \qquad \text{(VII.38)}$$

$$h_{[\mu\nu]} = A_{\mu,\nu} - A_{\nu,\mu}. \qquad \text{(VII.39)}$$

If we use (VII.38), (VII.39) and separate the symmetric and the skew part of (VII.32), the field equations become

$$k_1\,\Box\,\varphi^{\alpha\beta} = 2\,T_{D0}^{<\alpha\beta>} \qquad \text{(VII.40)}$$

$$\frac{1}{2}k_2\,\varepsilon^{\alpha\beta\varrho\sigma}\,\Box\,h_{[\varrho\sigma]} = ik_2\,\Box\,\check{h}^{[\alpha\beta]} = -T_{D0}^{[\alpha\beta]}. \qquad \text{(VII.41)}$$

It is well known[(9)] that the antisymmetric part of $T_{D0}^{\alpha\beta}$ is related to the spin angular-momentum tensor $\mathfrak{S}^{\alpha\beta\gamma}$ of a Dirac particle. In fact

$$T^{D[\alpha\beta]}_{0} = \frac{1}{2}\mathfrak{S}^{\alpha\beta\gamma}{}_{,\gamma}$$

and (VII.40), (VII.41) tell us that while the symmetric part of the field is coupled to the symmetric energy-momentum tensor of matter, the antisymmetric part is coupled to the spin angular momentum.

The solution of equations (VII.40), (VII.41) may give us some information about the constant k_2, but it is clear that a meaningful solution can be obtained only within the framework of a quantized theory.

To conclude this section we want to add the remark that equations similar to (VII.27), i. e. with the same coupling term $\alpha^{\mu}\,\Phi_{\mu}$, are valid also for the neutrino field. Using the two-spinor formalism (see appendix B), we can write the neutrino Lagrangian as

$$\mathfrak{L}_{\nu} = \frac{i}{2}\hat{g}\left\{\psi_A\,g^{\mu AB}\,\psi_{B,\mu} - \psi_{A,\mu}\,g^{\mu AB}\,\psi_B\right\}. \qquad \text{(VII.42)}$$

Again $\psi_{A,\mu}$ is simply the partial derivative.

Performing the variation with respect to $\psi_{\dot{A}}$ and ψ_B and using (VII.28), we obtain from (VII.42) the equations

$$\left\{\begin{array}{l} g^{\mu \dot{A} B}\, \psi_{B,\,\mu} - \frac{1}{2}\, g^{\mu \dot{A} B}\, \Phi_\mu\, \psi_B = 0 \\ \psi_{\dot{A},\,\mu}\, g^{\mu \dot{A} B} - \frac{1}{2}\, \psi_{\dot{A}}\, g^{\mu \dot{A} B}\, \Phi_\mu = 0 \end{array}\right\} \qquad \text{(VII.43)}$$

and the continuity equation

$$J^\mu{}_{;\,\mu} = (\psi_{\dot{A}}\, g^{\mu \dot{A} B}\, \psi_B)_{;\,\mu} = 0 . \qquad \text{(VII.44)}$$

VIII. Solution of the Field Equations for a Static, Spherically Symmetric System

In the case of a static, spherically symmetric system, the field equations are

$$k_1\, G^{\alpha\beta} + k_2\, F^{<\alpha\beta>} = -\, T^{(b)\,\alpha\beta} \qquad \text{(VIII.1)}$$

$$k_2\, F^{[\alpha\beta]} = 0 . \qquad \text{(VIII.2)}$$

Let us introduce an isotropic coordinate system, where the line element is of the type

$$ds^2 = b\,(r)\,(dx^0)^2 - a\,(r) \sum_1^3{}_\alpha\, (dx^\alpha)^2 \qquad \text{(VIII.3)}$$

and

$$r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 .$$

A solution of (VIII.2), satisfying also the relation (II.5), is then given by

$$g^{\alpha}_{\hat{\alpha}} = \frac{\delta^{\alpha}_{\hat{\alpha}}}{\sqrt{\varepsilon_{\alpha)}\, g_{\alpha\alpha}}}, \qquad \text{(VIII.4)}$$

where $\varepsilon_\alpha = (1, -1, -1, -1)$ and the bracket after the index α means that no summation over α should be performed.

In fact, using (VIII.4) and the notation

$$A' = \frac{dA\,(r)}{ds}, \quad n_\alpha = \frac{\partial r}{\partial x^\alpha} = \left(0, \frac{x^i}{r}\right),$$

we have

$$\left.\begin{aligned} \Lambda_{[\alpha\beta]}{}^{\gamma} &= \frac{1}{2}\,[ln\,(\varepsilon_{\alpha)}\,g_{\alpha\alpha})]'\,\delta^{\gamma}_{\alpha}\,n^{\beta} \\ &\quad -\frac{1}{2}\,[ln\,(\varepsilon_{\beta)}\,g_{\beta\beta})]'\,\delta^{\gamma}_{\beta}\,n_{\alpha}, \end{aligned}\right\} \quad \text{(VIII.5)}$$

$$\Phi_{\alpha} = -\,[ln\,a\,\sqrt{b}]'\,n_{\alpha}. \quad \text{(VIII.6)}$$

From (VIII.5), (VIII.6) it follows that

$$\check{\Lambda}_{\alpha} = 0, \quad \text{(VIII.7)}$$

$$\Phi_{\alpha,\,\beta} - \Phi_{\beta,\,\alpha} = 0. \quad \text{(VIII.8)}$$

When we use this last result, the tensor $F^{\alpha\beta}$ becomes simply

$$F^{\alpha\beta} = 2\,\eta^{\alpha\varrho\sigma\tau}\,\Phi_{\varrho}\,\Lambda_{[\sigma\tau]}{}^{\beta} + \frac{1}{2}\,g^{\alpha\beta}\,\eta^{\varrho\sigma\lambda\mu}\,\Lambda_{[\varrho\sigma]\,\nu}\,\Lambda_{[\lambda\mu]}{}^{\nu},$$

and by means of (VIII.5), (VIII.6) we have

$$\left.\begin{aligned} F^{\alpha\beta} &= -\,\eta^{\alpha\varrho\sigma\tau}\,(ln\,a\,\sqrt{b})'\,\{[ln\,(\varepsilon_{\sigma)}\,g_{\sigma\sigma})]'\,\delta^{\beta}_{\sigma}\,n_{\tau} \\ &\quad -[ln\,(\varepsilon_{\tau)}\,g_{\tau\tau})]'\,\delta^{\beta}_{\tau}\,n_{\sigma}\}\,n_{\varrho} \\ &\quad +\frac{1}{4}\,g^{\alpha\beta}\,\eta^{\varrho\sigma\lambda\mu}\,\{[ln\,(\varepsilon_{\varrho)}\,g_{\varrho\varrho})]'\,g_{\varrho\nu}\,n_{\sigma} \\ &-[ln\,(\varepsilon_{\sigma)}\,g_{\sigma\sigma})]'\,g_{\sigma\nu}\,n_{\varrho}\}\{[ln\,(\varepsilon_{\lambda)}\,g_{\lambda\lambda})]'\,\delta^{\nu}_{\lambda}\,n_{\mu} - [ln\,(\varepsilon_{\mu)}\,g_{\mu\mu})]'\,\delta^{\nu}_{\mu}\,n_{\lambda}\} \equiv 0. \end{aligned}\right\} \quad \text{(VIII.9)}$$

Since the tensor $F^{\alpha\beta}$ vanishes, we are left with the equation

$$k_1\,G_{\alpha\beta} = -\,T^{(b)}_{\alpha\beta}, \quad \text{(VIII.10)}$$

which determines in the usual way the two functions $a(r)$, $b(r)$.

The fact that the Schwarzschild metric holds for a static, spherically symmetric system is of course very important, since it allows us to say that this theory predicts correctly the results of the three experimental tests of the theory of general relativity.

IX. The Energy-Momentum Conservation Law

In section III the conservation law

$$g^{\hat{}}_{\cdot}\,\Theta^{\lambda}_{\hat{\lambda};\,\lambda} = 0 \quad \text{(IX.1)}$$

was obtained, and it was further shown that a superpotential $g^{\hat{}}_{\cdot}\,U^{[\lambda\delta]}_{\hat{\lambda}}$ exists such that

$$\Theta^{\lambda}_{\hat{\lambda}} = U^{[\lambda\delta]}_{\hat{\lambda};\,\delta}. \quad \text{(IX.2)}$$

From (A.11), (A.20) it is easy to see that

$$\left.\begin{aligned} U_{\hat{\lambda}}^{[\lambda\mu]} &= k_1\left\{g_{\hat{\lambda}\sigma}\,\gamma^{\lambda\mu\sigma} + g_{\hat{\lambda}}{}^{\mu}\,\Phi^{\lambda} - g_{\hat{\lambda}}^{\lambda}\,\Phi^{\mu}\right\} \\ &+ k_2\left\{\check{\Lambda}^{\mu}\, g_{\hat{\lambda}}^{\lambda} - \check{\Lambda}^{\lambda}\, g_{\hat{\lambda}}^{\mu} - 2\,\eta^{\lambda\mu\alpha\beta}\,\Phi_{\alpha}\, g_{\hat{\lambda}\beta}\right\}. \end{aligned}\right\} \tag{IX.3}$$

In the case of a static closed system, and in the absence of fermions, the metric is asymptotically of the Schwarzschild type, and the gravitational part of $\Theta_{\hat{\lambda}}^{\lambda}$, which is bilinear in the first derivatives of $g_{\hat{\lambda}}^{\lambda}$, behaves at infinity like $1/r^4$. Hence the four quantities

$$P_{\hat{\lambda}} = \int_{x_0 = \text{const}} g^{\hat{}}\, \Theta_{\hat{\lambda}}^{0}\, d^3 x \tag{IX.4}$$

are constant in time and invariant under coordinate transformations. We can write (IX.4) in a manifestly covariant form, substituting for the surface $x_0 = \text{const}$ a general three-dimensional, time-like surface Σ, as

$$P_{\hat{\lambda}} = \int_{\Sigma} \Theta_{\hat{\lambda}}^{\lambda}\, n_{\lambda}\, d\Sigma, \tag{IX.5}$$

where n_{λ} is the unit vector orthogonal to Σ, and $d\Sigma$ is the invariant volume element.

Using (IX.2), we can also express $P_{\hat{\lambda}}$ as an integral over a two-dimensional, space-like surface S, the boundary of $x_0 = \text{const}$:

$$P_{\hat{\lambda}} = \int_S g^{\hat{}}\, U_{\hat{\lambda}}^{[0k]}\, n_k\, dS, \tag{IX.6}$$

where n_k is a unit three-vector orthogonal to the surface element dS.

The relation between the four scalars $P_{\hat{\lambda}}$ and the total energy and momentum can easily be seen in the case of a closed system. We now want to introduce a conservation law of the usual form. To do this we introduce the tensor density

$$\mathfrak{U}_{\nu}^{[\lambda\mu]} = g^{\hat{}}\, g_{\nu}^{\hat{\lambda}}\, U_{\hat{\lambda}}^{[\lambda\mu]} \tag{IX.7}$$

and the quantity

$$\mathrm{T}_{\nu}{}^{\lambda} = \mathfrak{U}_{\nu}^{[\lambda\mu]}{}_{,\mu}. \tag{IX.8}$$

Clearly $\mathrm{T}_{\nu}{}^{\lambda}$ satisfies the conservation law

$$\mathrm{T}_{\nu}{}^{\lambda}{}_{,\lambda} = 0. \tag{IX.9}$$

From (IX.3), (IX.7) it follows that

$$\mathfrak{U}_{\nu}^{[\lambda\mu]} = \mathfrak{U}_{1\nu}^{[\lambda\mu]} + \mathfrak{U}_{2\nu}^{[\lambda\mu]}, \tag{IX.10}$$

where

$$\mathfrak{U}_{1\nu}^{[\lambda\mu]} = k_1\, g^{\hat{:}}\, [\gamma_\nu^{\lambda\mu} + \delta_\nu^\mu\, \Phi^\lambda - \delta_\nu^\lambda\, \Phi^\mu] \tag{IX.11}$$

is the Møller superpotential[5], and

$$\mathfrak{U}_{2\nu}^{[\lambda\mu]} = k_2\, g^{\hat{:}}\, [\check{\Lambda}^\mu\, \delta_\nu^\lambda - \check{\Lambda}^\lambda\, \delta_\nu^\mu - 2\, \eta^{\lambda\mu\alpha\beta}\, \Phi_\alpha\, g_{\beta\nu}]\,. \tag{IX.12}$$

In the case of a static, spherically symmetric system and in isotropic coordinates the Møller superpotential is given by[5]

$$\mathfrak{U}_{1\nu}^{[\lambda\mu]} = k_1 \sqrt{ab} \left[ln \frac{a\sqrt{b}}{\sqrt{|g_{\mu\mu}|}} \right]' (\delta_\nu^\lambda\, n^\mu - \delta_\nu^\mu\, n^\lambda), \tag{IX.13}$$

where the notations of section VIII have been used. For the second term of $\mathfrak{U}_\nu^{[\lambda\mu]}$ we obtain

$$\mathfrak{U}_{2\nu}^{[\lambda\mu]} = k_2\, \eta^{\alpha\beta\lambda\mu}\, [ln\, a\sqrt{b}]'\, g_{\beta\nu}\, n_\alpha \tag{IX.14}$$

and, as an immediate consequence,

$$\mathfrak{U}_{2\nu\ ,\mu}^{[\lambda\mu]} = 0\,. \tag{IX.15}$$

The quantity $\mathsf{T}_\nu{}^\lambda$ is now given by

$$\mathsf{T}_\nu{}^\lambda = \mathfrak{U}_{1\nu\ ,\mu}^{[\lambda\mu]} \tag{IX.16}$$

and is equal to the Møller energy-momentum complex. It is worth remembering that in isotropic coordinates the Møller complex $\mathsf{T}_\nu^{M\,\lambda}$ is equal to the Einstein complex $\Theta_\nu^{E\,\lambda}$ so that we have in this case

$$\mathsf{T}_\nu{}^\lambda = \mathsf{T}^M{}_\nu{}^\lambda = \Theta^E{}_\nu{}^\lambda. \tag{IX.17}$$

This result allows us to identify $\mathsf{T}_\nu{}^\lambda$, defined by (IX.8), as the energy-momentum complex.

We can now introduce the total energy and momentum for a closed system

$$P_\nu = \int_{x_0 = \mathrm{const}} \mathsf{T}_\nu{}^0\, d^3 x\,. \tag{IX.18}$$

The relation between P_ν and $P_{\hat{\nu}}$ can be found if we write P_ν in the form

$$P_\nu = \int_S \mathfrak{U}_\nu^{0i}\, n_i\, dS, \tag{IX.19}$$

where n_i, dS and S have the same meaning as in (IX.6). Then, using the fact that, in isotropic coordinates and for $r \to \infty$, $\mathfrak{U}_{1\nu}^{0i} \sim 0\left(\frac{1}{r^2}\right)$ and $g_\nu^{\hat{\lambda}} = \delta_\nu^{\hat{\lambda}} + 0\left(\frac{1}{r}\right)$, and further that $\int_S \mathfrak{U}_{2\nu}^{0i}\, n_i\, dS = 0$, we have

$$\left.\begin{aligned} P_\nu &= \int_S \mathfrak{U}_\nu^{[0i]}\, n_i\, dS = \int_S g^{\hat{:}}\, g_\nu^{\hat{\lambda}}\, U_{\hat{\lambda}}^{[0i]}\, n_i\, dS \\ &= \int_S g^{\hat{:}}\, \delta_\nu^{\hat{\lambda}}\, U_{\hat{\lambda}}^{[0i]}\, n_i\, dS = \delta_\nu^{\hat{\lambda}}\, P_{\hat{\lambda}}. \end{aligned}\right\} \quad (IX.20)$$

The superpotential defined by (IX.7) is a true tensor density; hence $\mathfrak{U}_0{}^{[\lambda\mu]}$ is transformed like an antisymmetric tensor under the group of purely spatial transformations

$$\left\{\begin{aligned} x'^0 &= x^0 \\ x'^i &= x'^i(x^k). \end{aligned}\right\} \quad (IX.21)$$

Then $\mathfrak{U}_0{}^{[\lambda\mu]}{}_{,\mu}$ will be transformed like a vector density under the same group, and in particular $\mathfrak{U}_0{}^{[0\mu]}{}_{,\mu}$ will be a scalar so that it is possible to give to $\mathsf{T}_0{}^0 = \mathfrak{U}_0{}^{[0\mu]}{}_{,\mu}$ the meaning of an energy density.

In contrast to $\Theta_{\hat{\lambda}}^{\hat{\lambda}}$, $\mathsf{T}_\nu{}^\lambda$ is not a tensor. On the other hand $\mathsf{T}_\nu{}^\lambda$ is a scalar under the "gauge" group of the constant tetrad rotations, while $\Theta_{\hat{\lambda}}^{\lambda}$ is a vector with respect to that group. The same is true of P_ν and $P_{\hat{\nu}}$. The situation does not lead to difficulties for the total energy and momentum (IX.4), since the tetrad rotations can be interpreted, for $r \to \infty$, as Lorentz transformations. In fact, to write the total energy momentum in the form (IX.6) is a compact way of saying that this quantity is transformed like a four-vector under Lorentz transformations.

When we consider the local properties of the field instead of the global quantity $P_{\hat{\lambda}}$, we are not allowed to use $\Theta_{\hat{\lambda}}^{\lambda}$ and $\mathsf{T}_\nu{}^\lambda$ in the same way. In particular, $\mathsf{T}_0{}^0$ and not $\Theta_{\hat{0}}^0$ is the energy density, since it is not possible in general to give to the tetrad rotations the meaning of Lorentz transformations.

The explicit form of $\mathsf{T}_\nu{}^\lambda$ can be evaluated from its definition (IX.8):

$$\left.\begin{aligned} \mathsf{T}_\nu{}^\lambda &= \mathfrak{U}_\nu^{[\lambda\mu]}{}_{,\mu} = \left(g^{\hat{:}}\, g_\nu^{\hat{\lambda}}\, U_{\hat{\lambda}}^{[\lambda\mu]}\right)_{,\mu} \\ &= g_\nu^{\hat{\lambda}}\, g^{\hat{:}}\, U_{\hat{\lambda}\,;\mu}^{[\lambda\mu]} + g^{\hat{:}}\, U_{\hat{\lambda}}^{[\lambda\mu]}\, g_{\nu,\mu}^{\hat{\lambda}} \\ &= g^{\hat{:}}\, \Theta_\nu{}^\lambda + \mathfrak{U}_\varrho^{[\lambda\mu]}\, \Delta_{\nu\mu}^\varrho. \end{aligned}\right\} \quad (IX.22)$$

Since

$$\Theta_\nu{}^\lambda = g_\nu{}^{\hat{\lambda}}\left\{k_1\, U_{1\hat{\lambda}}^{\lambda} + k_2\, U_{2\hat{\lambda}}^{\lambda} + T^{(b)}{}_{\hat{\lambda}}^{\lambda} + T^{(f)}{}_{\hat{\lambda}}^{\lambda}\right\}, \quad (IX.23)$$

we have from (A.14), (A.21), (IX.11), (IX.12), (IX.23)

$$\mathsf{T}_\nu{}^\lambda = g^{\hat{:}}\, [t_\nu{}^\lambda + T^{(b)}{}_\nu{}^\lambda + T^{(f)}{}_\nu{}^\lambda] \quad (IX.24)$$

$$\left.\begin{aligned} t_\nu{}^\lambda = k_1 \Big[& \gamma^{\lambda\varrho\sigma} \Lambda_{[\varrho\nu]\sigma} + \Phi^\sigma \Lambda_{[\nu\sigma]}{}^\lambda \\ & - \Phi_\nu \Phi^\lambda - \frac{1}{2} g^{\cdot}_{\cdot} \delta_\nu{}^\lambda \mathfrak{L}_M + U_1{}_\varrho^{[\lambda\mu]} \Delta^\varrho{}_{\nu\mu} \Big] \\ + k_2 [& \check{\Lambda}^\varrho \Lambda_{[\varrho\nu]}{}^\lambda - \eta^{\alpha\beta\gamma\lambda} \Phi_\gamma \Lambda_{[\alpha\beta]\nu} + U_2{}_\varrho^{[\lambda\mu]} \Delta^\varrho{}_{\nu\mu}] . \end{aligned}\right\} \quad \text{(IX.25)}$$

The form (IX.22) of $\mathsf{T}_\nu{}^\lambda$, which was already derived by MØLLER [(5)] for his complex, allows us to establish the transformation properties of our complex. In fact they are the same as for $\mathsf{T}^M{}_\nu{}^\lambda$, namely

$$t'^{\beta}_{\alpha} = \frac{\partial x'^\beta}{\partial x^\mu} \left\{ \frac{\partial x^\lambda}{\partial x'^\alpha} t^\mu_\lambda + \frac{\partial}{\partial x^\nu} \left(\frac{\partial x^\varrho}{\partial x'^\alpha} \right) \mathfrak{U}_\varrho^{[\mu\nu]} \right\}. \quad \text{(IX.26)}$$

Now we want to examine our energy-momentum complex in the weak-field case, and for simplicity we will assume that the coordinate condition (V.22) holds. Using again the expression (V.1) for $g^{\alpha}_{\hat{\alpha}}$, we have

$$\gamma_{\alpha\beta}{}^\gamma = \frac{1}{2} (h_{<\beta^{\gamma}_{\cdot}>, \alpha} - h_{<\alpha^{\gamma}_{\cdot}>, \beta})$$

$$\Phi_\alpha = -\frac{1}{4} h_{,\alpha}$$

$$h_{[\alpha\beta]} = 0,$$

so that

$$\left.\begin{aligned} \mathfrak{U}_\nu^{[\varrho\sigma]} = \frac{1}{2} k_1 \Big\{ & h_{<\nu^{\sigma}_{\cdot}>}{}^{,\varrho} - h_{<\nu^{\varrho}_{\cdot}>}{}^{,\sigma} \\ + \frac{1}{2} (\delta^\varrho_\nu h^{,\sigma} - \delta^\sigma_\nu h^{,\varrho}) \Big\} & + \frac{1}{2} k_2 \, \varepsilon^{\varrho\sigma\alpha\beta} \eta_{\beta\nu} h_{,\alpha} . \end{aligned}\right\} \quad \text{(IX.27)}$$

From (IX.8) we find, using the coordinate condition,

$$\mathsf{T}_\nu{}^\lambda = -\frac{1}{2} k_1 \Box \varphi_\nu{}^\lambda = \mathsf{T}^{(b)}_\nu{}^\lambda, \quad \text{(IX.28)}$$

so that $\mathsf{T}_\nu{}^\lambda$ is equal to the matter tensor, in agreement with the fact that the gravitational complex is of the second order.

If we perform the evaluation of $t_\alpha{}^\beta$ to the second order, using the $h_{<\alpha\beta>}$ as determined from the first-order approximation, the tensor

$$t_\alpha{}^\beta = t^M{}_\alpha{}^\beta + t'_\alpha{}^\beta \quad \text{(IX.29)}$$

with

$$\left.\begin{aligned} t^{M}{}_{\alpha}{}^{\beta} = \frac{1}{4} k_1 \Big\{ & (h_{<\lambda^{\mu}_{\cdot}>}{}^{,\beta} - h_{<\lambda^{\beta}_{\cdot}>}{}^{,\mu})\, h^{<\lambda^{\cdot}_{\mu}>}{}_{,\alpha} \\ & + \frac{1}{2} (h_{,\lambda}\, h^{<\lambda\beta>}{}_{,\alpha} - h_{,\alpha}\, h^{,\beta}) - \delta_{\alpha}{}^{\beta}\, \mathfrak{L}_M \Big\} \end{aligned} \right\} \quad \text{(IX.30)}$$

$$\mathfrak{L}_M = \frac{1}{4} h_{<\alpha\beta>,\gamma}\, h^{<\alpha\beta>,\gamma} - \frac{1}{8} h_{,\alpha}\, h^{,\alpha} \quad \text{(IX.31)}$$

and

$$t'_{\alpha}{}^{\beta} = X_{\alpha}{}^{[\beta\lambda]}{}_{,\lambda} + X^{\beta}{}^{[\cdot\lambda]}_{\alpha}{}_{,\lambda} \quad \text{(IX.32)}$$

$$X_{\alpha}{}^{[\beta\lambda]} = \frac{1}{4} k_2\, \varepsilon^{\beta\mu\lambda\nu}\, h\, h_{<\nu\alpha>,\mu} \quad \text{(IX.33)}$$

is obtained.

The tensor $t^{M}{}_{\alpha}{}^{\beta}$ is just the Møller weak-field gravitational complex. It is noteworthy that the symmetric term $t'_{\alpha}{}^{\beta}$ can be written as a divergence.

We want to thank Professor C. Møller for many illuminating discussions. One of us (C. P.) wants to express his gratitude to Professor C. Møller for the generous hospitality accorded to him by NORDITA.

Appendix A

The Einstein Tensor, the Tensor $F^{\alpha\beta}$ and the Evaluation of the Superpotential

We will recall here the connection, first discovered by Møller [(4)] [(5)], between the tetrads and the curvature tensor.

Replacing in the basic formula

$$A_{\beta;\gamma;\delta} - A_{\beta;\delta;\gamma} = -A_{\alpha} R^{\alpha}{}_{\beta\gamma\delta}$$

the vector A_{α} by $g^{\hat{\alpha}}_{\alpha}$ and multiplying by $g_{\hat{\alpha}}^{\varrho}$, we get

$$R^{\varrho}{}_{\beta\gamma\delta} = -g^{\varrho}_{\hat{\alpha}} \left(g^{\hat{\alpha}}{}_{\beta;\gamma;\delta} - g^{\hat{\alpha}}{}_{\beta;\delta;\gamma} \right). \quad \text{(A.1)}$$

By contracting this we obtain

$$R_{\alpha\beta} = R^{\varrho}_{\alpha\varrho\beta} = -g^{\varrho}_{\hat{\alpha}} \left(g^{\hat{\alpha}}_{\alpha;\varrho;\beta} - g^{\hat{\alpha}}_{\alpha;\beta;\varrho} \right), \quad \text{(A.2)}$$

$$R = R^{\alpha}_{\alpha} = -g^{\alpha\beta}\, g^{\varrho}_{\hat{\alpha}} \left(g^{\hat{\alpha}}_{\alpha;\varrho;\beta} - g^{\hat{\alpha}}_{\alpha;\beta;\varrho} \right). \quad \text{(A.3)}$$

The last formula can easily be rewritten as

$$
\left.\begin{aligned}
g\hat{:} R &= \left[g\hat{:}\left(g_{\hat{\varrho}}^{\sigma}\, g^{\hat{\varrho}\varrho}{}_{;\varrho} - g_{\hat{\varrho}}^{\varrho}\, g^{\hat{\varrho}\sigma}{}_{;\varrho}\right)\right]_{,\sigma}\\
&+ g\hat{:}\left(g_{\hat{\varrho}}{}^{\varrho}{}_{;\sigma}\, g^{\hat{\varrho}\sigma}{}_{;\varrho} - g_{\hat{\varrho}}{}^{\varrho}{}_{;\varrho}\, g^{\hat{\varrho}\sigma}{}_{;\sigma}\right)\\
&= -2\left(g\hat{:}\,\Phi^{\sigma}\right)_{,\sigma} + g\hat{:}\left(\gamma_{\tau\varrho\sigma}\gamma^{\tau\sigma\varrho} - \Phi_{\varrho}\Phi^{\varrho}\right)\\
&\equiv H^{\sigma}{}_{,\sigma} + \mathfrak{L}_{M}.
\end{aligned}\right\}\qquad (A.4)
$$

The importance of this decomposition, as compared with the usual one, which can for example be written as[11]

$$
\sqrt{-g}\,R = -\left\{\frac{1}{\sqrt{-g}}\left(-g g^{\varrho\sigma}\right)_{,\sigma}\right\}_{,\varrho} + \sqrt{-g}\, g^{\alpha\beta}\left\{\Gamma^{\varrho}_{\alpha\sigma}\Gamma^{\sigma}_{\varrho\beta} - \Gamma^{\varrho}_{\alpha\beta}\Gamma^{\sigma}_{\varrho\sigma}\right\},
$$

lies in the fact that $\mathfrak{L}_M$ is a scalar with respect to coordinate transformations, so that the action principle is of the type used in other field theories.

By means of (A.2), (A.4) the Ricci tensor can be written as

$$
\left.\begin{aligned}
G_{\alpha\beta} &= g_{\hat{\alpha}}^{\varrho}\left(g^{\hat{\alpha}}_{\alpha;\beta;\varrho} - g^{\hat{\alpha}}_{\alpha;\varrho;\beta}\right)\\
&-\frac{1}{2} g_{\alpha\beta}\, g_{\hat{\alpha}}^{\varrho}\left(g^{\hat{\alpha}\sigma}{}_{;\sigma;\varrho} - g^{\hat{\alpha}\sigma}{}_{;\varrho;\sigma}\right),
\end{aligned}\right\}\qquad (A.5)
$$

and, multiplying by $g_{\hat{\beta}\alpha}$, we have

$$
\left.\begin{aligned}
G_{\hat{\beta}}^{\beta} &= g_{\hat{\beta}\alpha}\, G^{\alpha\beta} = g_{\hat{\beta}\alpha}\, g_{\hat{\alpha}}^{\varrho}\left(g^{\hat{\alpha}\alpha;\beta}{}_{;\varrho} - g^{\hat{\alpha}\alpha}{}_{;\varrho}{}^{;\beta}\right)\\
&-\frac{1}{2} g_{\hat{\beta}}^{\beta}\, g_{\hat{\alpha}}^{\varrho}\left(g^{\hat{\alpha}\sigma}{}_{;\sigma;\varrho} - g^{\hat{\alpha}\sigma}{}_{;\varrho;\sigma}\right).
\end{aligned}\right\}\qquad (A.6)
$$

The mixed expression $G_{\hat{\beta}}^{\beta}$ can be decomposed in the form

$$
G_{\hat{\beta}}^{\beta} = -\mathfrak{U}_{1}{}^{[\beta\gamma]}_{\hat{\beta}}{}_{;\gamma} + \mathfrak{U}_{1}{}_{\hat{\beta}}^{\beta}, \qquad (A.7)
$$

where $\mathfrak{U}_{1}{}_{\hat{\beta}}^{\beta}$ is bilinear and $\mathfrak{U}_{1}{}^{[\beta\gamma]}_{\hat{\beta}}$ linear in the first derivatives of $g_{\hat{\alpha}}^{\alpha}$.

To establish the form (A.7) of $G_{\hat{\beta}}^{\beta}$ it is better to start from the alternative expression of $R_{\alpha\beta}$ and R given by Møller[5]:

$$
R^{\alpha\beta} = \gamma^{\varrho\alpha\beta}/_{\varrho} - \Phi^{\alpha/\beta} - \gamma^{\varrho\alpha\beta}\Phi_{\varrho} + \gamma^{\varrho\alpha}{}_{\sigma}\gamma^{\sigma\beta}{}_{\varrho}, \qquad (A.8)
$$

$$
R = -2\,\Phi^{\varrho}/_{\varrho} + \gamma_{\varrho\sigma\tau}\gamma^{\varrho\tau\sigma} - \Phi_{\varrho}\Phi^{\varrho}. \qquad (A.9)
$$

Using the fact that the absolute derivative $g^{\hat{\alpha}}_{\alpha/\beta} = 0$, we obtain

$$\left.\begin{aligned} G_{\hat{\beta}}^{\ \beta} &= g_{\hat{\beta}\alpha} G^{\alpha\beta} = g_{\hat{\beta}\alpha} G^{\beta\alpha} \\ &= \left(g_{\hat{\beta}\alpha}\gamma^{\varrho\beta\alpha} + g^{\beta}_{\hat{\beta}}\Phi^{\varrho} - g^{\ \varrho}_{\hat{\beta}}\Phi^{\beta}\right)_{/\varrho} + g_{\hat{\beta}\alpha}\left(\gamma^{\varrho\beta}{}_{\sigma}\gamma^{\sigma\alpha}{}_{\varrho}\right. \\ &\quad \left. - \Phi_{\varrho}\gamma^{\varrho\beta\alpha}\right) - \frac{1}{2} g^{\beta}_{\hat{\beta}}\left(\gamma_{\varrho\sigma\tau}\gamma^{\varrho\tau\sigma} + \Phi_{\varrho}\Phi^{\varrho}\right). \end{aligned}\right\} \quad \text{(A.10)}$$

Defining

$$U_{1\hat{\beta}}^{[\beta\varrho]} = g_{\hat{\beta}\alpha}\gamma^{\beta\varrho\alpha} + g_{\hat{\beta}}^{\ \varrho}\Phi^{\beta} - g^{\beta}_{\hat{\beta}}\Phi^{\varrho} \quad \text{(A.11)}$$

and using the connection between the absolute and the covariant derivative[5]

$$U_{1\hat{\beta}/\varrho}^{[\beta\varrho]} = U_{1\hat{\beta};\varrho}^{[\beta\varrho]} + U_{1\hat{\beta}}^{[\sigma\beta]}\gamma^{\beta}_{\sigma\varrho} + U_{1\hat{\beta}}^{[\beta\sigma]}\gamma^{\varrho}_{\sigma\varrho}, \quad \text{(A.12)}$$

we easily find from (A.10), (A.11), (A.12)

$$\left.\begin{aligned} G_{\hat{\beta}}^{\ \beta} &= -U_{1\hat{\beta};\varrho}^{[\beta\varrho]} - U_{1\hat{\beta}}^{[\sigma\varrho]}\gamma^{\beta}{}_{\sigma\varrho} + g^{\sigma}_{\hat{\beta}}\Phi^{\beta}\Phi_{\sigma} \\ &\quad + g_{\hat{\beta}\alpha}\gamma^{\varrho\beta}{}_{\sigma}\gamma^{\sigma\alpha}{}_{\varrho} - \frac{1}{2} g^{\beta}_{\hat{\beta}}\, g\hat{:}\, \mathfrak{L}_M \end{aligned}\right\} \quad \text{(A.13)}$$

and

$$U_{1\hat{\beta}}^{\beta} = -U_{1\hat{\beta}}^{[\sigma\varrho]}\gamma^{\beta}{}_{\sigma\varrho} - g^{\sigma}_{\hat{\beta}}\Phi_{\sigma}\Phi^{\beta} + g_{\hat{\beta}\alpha}\gamma^{\varrho\beta}{}_{\sigma}\gamma^{\sigma\alpha}{}_{\varrho} - \frac{1}{2} g^{\beta}_{\hat{\beta}}\, g\hat{:}\, \mathfrak{L}_M. \quad \text{(A.14)}$$

A decomposition similar to (A.7) can be found for

$$g\hat{:}\, F^{\varrho}_{\hat{\varrho}} = \frac{\delta \check{I}_3}{\delta g^{\varrho}_{\hat{\varrho}}}. \quad \text{(A.15)}$$

The term $\check{I}_3$ of our Lagrangian can be written as

$$\left.\begin{aligned} \check{I}_3 &= -\frac{1}{2} g\hat{:} \left\{\eta^{\alpha\lambda\nu\mu} g^{\beta}_{\hat{\gamma}} - \eta^{\beta\lambda\nu\mu} g^{\alpha}_{\hat{\gamma}}\right\} g_{\hat{\mu}\mu} \\ &\quad \cdot \left(g^{\hat{\gamma}}_{\alpha,\beta} - g^{\hat{\gamma}}_{\beta,\alpha}\right)\left(g^{\hat{\mu}}_{\lambda,\nu} - g^{\hat{\mu}}_{\nu,\lambda}\right). \end{aligned}\right\} \quad \text{(A.16)}$$

Performing the variation of $\check{I}_3$, we have

$$g\hat{:}\, F^{\varrho}_{\hat{\varrho}} = \frac{\delta \check{I}_3}{\delta g^{\varrho}_{\hat{\varrho}}} = -g\hat{:}\, U_{2\hat{\varrho};\sigma}^{[\varrho\sigma]} + g\hat{:}\, U_{2\hat{\varrho}}^{\ \varrho}, \quad \text{(A.17)}$$

$$\left.\begin{aligned} -g\hat{:}\, U_{2\hat{\varrho};\sigma}^{[\varrho\sigma]} &= \frac{1}{2}\left\{ g\hat{:}\left(\eta^{\alpha\lambda\nu\mu} g^{\beta}_{\hat{\gamma}} - \eta^{\beta\lambda\nu\mu} g^{\alpha}_{\hat{\gamma}}\right) g_{\hat{\mu}\mu} \right. \\ &\quad \left. \cdot \frac{\partial}{\partial g^{\hat{\varrho}}_{\varrho,\sigma}}\left[\left(g^{\hat{\gamma}}_{\alpha,\beta} - g^{\hat{\gamma}}_{\beta,\alpha}\right)\left(g^{\hat{\mu}}_{\lambda,\nu} - g^{\hat{\mu}}_{\nu,\lambda}\right)\right]\right\}_{,\sigma}, \end{aligned}\right\} \quad \text{(A.18)}$$

$$\left.\begin{aligned} \overset{\cdot}{g}\, U_{2\hat{\varrho}}^{\;\;\varrho} = -\frac{1}{2}\frac{\partial}{\partial g^{\hat{\varrho}}_{\varrho}} \left\{ \overset{\cdot}{g} \left(\eta^{\alpha\lambda\nu\mu}\, g^{\beta}_{\hat{\gamma}} - \eta^{\beta\lambda\nu\mu}\, g^{\alpha}_{\hat{\gamma}} \right) g_{\hat{\mu}\mu} \right\} \\ \cdot \left(g^{\hat{\gamma}}_{\alpha,\beta} - g^{\hat{\gamma}}_{\beta,\alpha} \right) \left(g^{\hat{\mu}}_{\lambda,\nu} - g^{\hat{\mu}}_{\nu,\lambda} \right). \end{aligned}\right\} \tag{A.19}$$

Evaluating (A.17), (A.18), we obtain

$$-U_{2\hat{\varrho}}^{[\varrho\sigma]} = \check{\Lambda}^{\varrho}\, g^{\sigma}_{\hat{\varrho}} - \check{\Lambda}^{\sigma}\, g^{\varrho}_{\hat{\varrho}} + 2\,\eta^{\varrho\sigma\alpha\beta}\,\Phi_{\alpha}\, g_{\hat{\varrho}\beta}, \tag{A.20}$$

$$U_{2\hat{\varrho}}^{\;\;\varrho} = \check{\Lambda}^{\alpha}\, \Lambda_{[\alpha\beta]}^{\;\;\;\;\varrho}\, g^{\beta}_{\hat{\varrho}} - \eta^{\alpha\lambda\nu\varrho}\,\Phi_{\alpha} \left(g_{\hat{\varrho}\lambda,\nu} - g_{\hat{\varrho}\nu,\lambda} \right). \tag{A.21}$$

From (A.17), (A.20), (A.21) it follows that

$$\left.\begin{aligned} F_{\hat{\varrho}}^{\;\varrho} = \left[\check{\Lambda}^{\varrho}\, g^{\sigma}_{\hat{\varrho}} - \check{\Lambda}^{\sigma}\, g^{\varrho}_{\hat{\varrho}} + 2\,\eta^{\varrho\sigma\alpha\beta}\,\Phi_{\alpha}\, g_{\hat{\varrho}\beta} \right]_{;\sigma} \\ + \check{\Lambda}^{\alpha}\, \Lambda_{[\alpha\beta]}^{\;\;\;\;\varrho}\, g^{\beta}_{\hat{\varrho}} - \eta^{\alpha\lambda\nu\varrho}\,\Phi_{\alpha} \left(g_{\hat{\varrho}\lambda,\nu} - g_{\hat{\varrho}\nu,\lambda} \right). \end{aligned}\right\} \tag{A.22}$$

We can now define the tensor

$$F^{\sigma\varrho} = g^{\hat{\varrho}\sigma}\, F_{\hat{\varrho}}^{\;\varrho}. \tag{A.23}$$

By means of (A.22) this is easily seen to be

$$\left.\begin{aligned} F^{\sigma\varrho} &= \left\{ \check{\Lambda}^{\varrho}\, g^{\tau\sigma} - \check{\Lambda}^{\tau}\, g^{\sigma\varrho} + 2\,\eta^{\tau\alpha\varrho\sigma}\,\Phi_{\alpha} \right\}_{;\tau} \\ &\quad - \left\{ \check{\Lambda}^{\varrho}\,\gamma^{\tau\sigma}{}_{\tau} - \check{\Lambda}^{\tau}\,\gamma^{\varrho\sigma}{}_{\tau} + 2\,\eta^{\varrho\tau\alpha\beta}\,\Phi_{\alpha}\,\gamma_{\beta}{}^{\sigma}{}_{\tau} \right\} \\ &\quad + \check{\Lambda}^{\alpha}\, \Lambda_{[\alpha\,\cdot\,]}{}^{\sigma\varrho} - \eta^{\alpha\lambda\nu\varrho}\,\Phi_{\alpha}\, \Lambda_{[\lambda\nu]}{}^{\sigma} \\ &= \check{\Lambda}^{\varrho;\sigma} - g^{\varrho\sigma}\,\check{\Lambda}^{\tau}{}_{;\tau} + 2\,\eta^{\varrho\sigma\alpha\beta}\,\Phi_{\alpha;\beta} \\ &\quad - \Phi^{\sigma}\,\check{\Lambda}^{\varrho} + \gamma^{\varrho\sigma}{}_{\tau}\,\check{\Lambda}^{\tau} + \eta^{\varrho\alpha\beta\gamma}\,\Phi_{\alpha}\,\Lambda_{[\beta\gamma]}{}^{\sigma} \\ &\quad + \check{\Lambda}^{\alpha}\,\Lambda_{[\alpha\,\cdot\,]}{}^{\sigma\varrho} - \eta^{\lambda\nu\mu\varrho}\,\Phi_{\mu}\,\Lambda_{[\lambda\nu]}{}^{\sigma}. \end{aligned}\right\} \tag{A.24}$$

The term $\check{\Lambda}^{\tau}{}_{;\tau}$ can be written in a form showing explicitly that it depends only on the first derivatives of the tetrad field. In fact

$$\left.\begin{aligned} \overset{\cdot}{g}\,\check{\Lambda}^{\tau}{}_{;\tau} &= \overset{\cdot}{g}\,\eta^{\tau\alpha\beta\gamma}\,\Lambda_{[\alpha\beta]\gamma;\tau} \\ &= 2\left[\overset{\cdot}{g}\,\eta^{\tau\alpha\beta\gamma}\, g^{\hat{\alpha}}_{\gamma}\, g_{\hat{\alpha}\alpha,\beta} \right]_{,\tau} \\ &= -2\,\varepsilon^{\tau\alpha\beta\gamma}\, g^{\hat{\alpha}}_{\gamma,\tau}\, g_{\hat{\alpha}\alpha,\beta} \\ &= \frac{1}{2}\,\overset{\cdot}{g}\,\eta^{\tau\alpha\beta\gamma}\,\Lambda_{[\gamma\tau]\delta}\,\Lambda_{[\alpha\beta]}{}^{\delta} \\ &= -\frac{1}{2}\,\check{I}_{4}. \end{aligned}\right\} \tag{A.25}$$

Appendix B

Tensors and Spinors

Here we want to discuss how the concepts of tensor and spinor can be introduced in our theory.

It is well known that in the special theory of relativity the definition of tensors can be given from two different points of view. One can define a vector as an object that is transformed like the coordinates under a Lorentz transformation, and a tensor as an object that is transformed like a product of vectors; or one can consider the representations of the Lorentz group and introduce in this way both tensors and spinors as quantities that are transformed in a well-defined way under a Lorentz transformation.

Clearly the second approach is more fundamental, since it allows us to introduce both spinors and tensors in a very natural way.

When we go from the special to the general theory of relativity, we can use only the first point of view. In consequence, the spinor fields are not, in the general case, on the same basis as tensor fields, which is quite unsatisfactory(6).

In the framework of our theory it seems possible instead to re-establish the connection between tensors and spinors, as representations of the Lorentz group, and the usual non-group-theoretical definition.

Let us consider a Riemannian four-space V_4 in which a coordinate system $x^0, \cdots, x^3$ has been introduced. In each point of this space we have a tetrad whose components are $g^{\alpha}_{\hat{\alpha}}(x)$. The representation of the group L of the proper tetrad rotations gives us quantities

$$T^{\hat{\alpha}\cdots}_{\hat{\beta}\cdots}$$

and

$$\psi^{A\cdots}_{B\cdots} \qquad (A, B = 1, 2)$$

which are tensors or spinors with respect to L. The connection between the "local" tensor $T^{\hat{\alpha}\cdots}_{\hat{\beta}\cdots}$ and a "world" tensor $T^{\alpha\cdots}_{\beta\cdots}$ (as defined from the first point of view) is easily established with the help of the $g^{\hat{\alpha}}_{\alpha}$:

$$T^{\alpha\cdots}_{\beta\cdots} = g^{\alpha}_{\hat{\alpha}}\, g^{\hat{\beta}}_{\beta} \cdots T^{\hat{\alpha}\cdots}_{\hat{\beta}\cdots}.$$

The spinors, which have no "roofed" index, are simply equivalent to "world" scalars, while the connection between the "local" Dirac matrices $\gamma^{\hat{\alpha}}$ and the "world" matrices is again given by $g^{\alpha}_{\hat{\alpha}}$:

$$\gamma^{\alpha} = g^{\alpha}_{\hat{\alpha}}\, \gamma^{\hat{\alpha}}.$$

From this point of view the constant tetrad rotations, which leave our theory invariant, play a part analogous to that of the Lorentz transformation of special relativity. In general there exists no relation between tetrad rotations and coordinate transformations, as there does between "local" tensors and spinors and "world" tensors. Only in the limit of flat space-time and when we use cartesian coordinates are tetrad rotations equivalent to Lorentz transformations so that we obtain the formalism of special relativity as a limiting case.

Since we need it in section VI, we will now review the essential steps of the introduction of spinors, at the same time establishing our notations.

In this paper we consider three types of transformation:

(a) coordinate transformations $x'^{\alpha} = x'^{\alpha}(x^{\beta})$ (group C);
(b) tetrad rotations $A'^{\hat{\alpha}} = L^{\hat{\alpha}}{}_{\hat{\beta}} A^{\hat{\beta}}$ (group L);
(c) linear or antilinear unimodular transformations in the two-dimensional complex spin space, induced by proper or improper tetrad rotations (group T).

Objects that are transformed appropriately with respect to all groups will be called tensors and spinors.

We remember that to the group L belong, in our theory, only the constant tetrad rotations.

A general tensor density $\mathfrak{T}^{\alpha\cdots\hat{\alpha}}_{\beta\cdots\hat{\beta}}$ is transformed like

$$\left.\begin{aligned} \mathfrak{T}'^{\alpha\cdots\hat{\alpha}}_{\beta\cdots\hat{\beta}} = \left\{\det\left|\frac{\partial x'^{\varrho}}{\partial x^{\sigma}}\right|\right\}^{w} \left\{\det\left| L^{\hat{\varrho}}{}_{\hat{\sigma}}\right|\right\}^{\hat{w}} \\ \frac{\partial x'^{\alpha}}{\partial x^{\lambda}}\cdots\frac{\partial x^{\mu}}{\partial x'^{\beta}}\cdots L^{\hat{\alpha}}{}_{\hat{\lambda}}\cdots L^{-1\hat{\mu}}{}_{\hat{\beta}}\cdots T^{\lambda\cdots\hat{\lambda}\cdots}_{\mu\cdots\hat{\lambda}\cdots}, \end{aligned}\right\} \tag{B.1}$$

where w, $\hat{w}$ are the weights of the (a) and (b) transformations, $L^{\hat{\alpha}}{}_{\hat{\lambda}}$ is submitted to the condition (II.7) and $L^{-1\hat{\mu}}{}_{\hat{\beta}}$ is defined by $L^{\hat{\alpha}}_{\hat{\varrho}} L^{-1\hat{\varrho}}{}_{\hat{\beta}} = \delta^{\hat{\alpha}}_{\hat{\beta}}$.* Note that $\det\left| L^{\hat{\alpha}}{}_{\hat{\beta}}\right|$ is equal to one in the case of proper tetrad rotations and to minus one in the case of improper rotations.

We give the name contravariant spinor to the quantity ψ^{A} $(A = 1, 2)$, which under linear, unimodular transformations $t^{A}{}_{B}$, $\det|t^{A}{}_{B}| = 1$, is transformed like**

$$\psi'^{A} = t^{A}{}_{B}\,\psi^{B}. \tag{B.2}$$

* From this definition and (II. 7) it follows that

$$L^{-1\hat{\alpha}}{}_{\hat{\varrho}} = g^{\hat{\alpha}\hat{\delta}} g_{\hat{\beta}\hat{\gamma}} L^{\hat{\gamma}}{}_{\hat{\delta}}.$$

** In order to simplify the notations we shall not here discuss spinor density. For a more general formulation see for instance the book by Corson (reference 9).

Spin indices may be raised or lowered with the help of the metrical spinors

$$g_{AB} = g^{AB} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix} \tag{B.3}$$

so that

$$\left.\begin{cases} \psi^A = \psi_B \, g^{BA} = -g^{AB} \, \psi_B \\ \psi_A = g_{AB} \, \psi^B . \end{cases}\right\} \tag{B.4}$$

The isomorphism between the groups L and T may be established with the help of the numerical spin matrices

$$g_{\hat{\alpha}\dot{A}B} = \left(\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & i \\ -i & 0 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \right). \tag{B.5}$$

We have, in the case of proper tetrad rotations,

$$L^{\hat{\alpha}}{}_{\hat{\beta}} = \frac{1}{2} g^{\hat{\alpha}\dot{A}B} \, g_{\hat{\beta}\dot{C}D} \, t^{\dot{C}}{}_{\dot{A}} \, t^{D}{}_{B}, \tag{B.6}$$

where

$$t^{\dot{A}}{}_{\dot{B}} = \left(t^{A}{}_{B} \right)^{*}$$

(the asterisk means complex conjugation).

The space of the spinors ψ_A is the space of the representation $D(1/2, 0)$ of the proper rotation group. The conjugate representation

$$\psi^{\dot{A}} = t^{\dot{A}}{}_{\dot{B}} \, \psi^{\dot{B}}, \quad \psi^{\dot{A}} = \left(\psi^A \right)^{*} \tag{B.7}$$

is transformed like the representation $D(0, 1/2)$.

Under improper Lorentz rotations the spinors are transformed by means of the antilinear operators $t^{A}{}_{\dot{B}}$, det $| t^{A}{}_{\dot{B}} | = 1$,

$$\psi^A = t^{A}{}_{\dot{B}} \, \psi^{\dot{B}}, \tag{B.8}$$

which mix the representations $D\left(\frac{1}{2}, 0\right)$ and $D\left(0, \frac{1}{2}\right)$. The isomorphism between the improper rotations and the antilinear operators $t^{A}{}_{\dot{B}}$ is given by

$$L^{\hat{\alpha}}{}_{\hat{\beta}} = \frac{1}{2} g^{\hat{\alpha}\dot{A}B} \, g_{\hat{\beta}\dot{C}D} \, t^{\dot{C}}{}_{B} \, t^{D}{}_{\dot{A}}. \tag{B.9}$$

The spin matrices $g_{\hat{\alpha}\dot{A}B}$ can be defined in a general way by the relations

$$\left(g^{\hat{\alpha}}{}_{\dot{A}B} \right)^{*} = g^{\hat{\alpha}}{}_{\dot{B}A}, \tag{B.10}$$

$$g_{\hat{\alpha}}{}^{\dot{B}A} \, g_{\hat{\beta}\dot{B}C} = g_{\hat{\alpha}\hat{\beta}} \, \delta^{A}{}_{C} + \frac{1}{2} \varepsilon_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} \, g^{\hat{\gamma}\dot{B}A} \, g^{\hat{\delta}}{}_{\dot{B}C}. \tag{B.11}$$

The four spinors ψ can be introduced as the direct sum of the two representations $D\left(\frac{1}{2},0\right)$, $D\left(0,\frac{1}{2}\right)$,

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \Phi^{\dot{1}} \\ \Phi^{\dot{2}} \end{pmatrix}.$$

The γ matrices can be given by the definition

$$\gamma^{\hat{\mu}\alpha}{}_{\beta} = \begin{Vmatrix} 0 & -i \left| g^{\hat{\mu}}{}_{\dot{B}A} \right| \\ i \left| g^{\hat{\mu}\dot{A}B} \right| & 0 \end{Vmatrix}$$

and satisfy the relation

$$\gamma^{\hat{\mu}}\gamma^{\hat{\nu}} + \gamma^{\hat{\nu}}\gamma^{\hat{\mu}} = 2\, g^{\hat{\mu}\hat{\nu}}.$$

Instead of the $\gamma^{\hat{\mu}}$ we will use the matrices $\alpha^{\hat{\mu}}$, β, which are all Hermitian, while $\gamma^{\hat{0}}$ is Hermitian and $\gamma^{\hat{1}}, \gamma^{\hat{2}}, \gamma^{\hat{3}}$ are anti-Hermitian(12). The relation between the γ and the α is

$$\gamma^{\hat{0}} = \beta$$

$$\beta\gamma^{\hat{i}} = \alpha^{\hat{i}} \qquad (i = 1, 2, 3)$$

$$\alpha^{\hat{0}} = 1\,.$$

Together with spinors we must define their derivatives. In contrast to what is the case in the usual theory, this can be done quite simply. In fact, since ψ_A is a scalar with respect to the group C, $\psi_{A,\mu}$ is transformed like a vector. Further, with respect to the group L $\psi_{A,\mu}$ is transformed like a spinor, as follows from the fact that only constant tetrad rotations belong to L. So we can simply define the derivative of a two- or four-spinor as $\psi_{A,\mu}$ or $\Psi_{,\mu}$, and there is no need to introduce additive terms, as we must do in the Einstein theory.

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